Numerical Integration of Parabolic Partial Differential Equations

In Fluid Mechanics we can frequently find Parabolic partial Differential equations. Boundary layer equations and Parabolized Navier Stokes equations, are only two significant examples of these type of equations.
In this work, I investigate several finite difference formulations of a given parabolic partial differential equation.

- *Parabolic partial differntial equation*

A typical second-order parabolic PDE can be written as:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

Where *u(y,t)* is the dependent function, *x* is a spatial coordinate (here we assume one space dimension) , *t* is time and ν is a constant.
The aim is to numerically solve eq. (1) through different solution schemes.

- FTCS (Forward time/Central Space) method

The FTCS scheme uses, for the approximation of the derivatives in eq.(1), a forward difference scheme in time and a central difference scheme in space.
This leads to the following approximation of the derivatives:

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta y)^2} + O(\Delta y^2)$$

Thus, the model differential equation becomes:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta y)^2}$$

- Laasonen method

When we approximate the derivatives as follows

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta y)^2} + O(\Delta y^2)$$

we obtain an implicit equation, since more than one unknown appears in a single equation.

This equation read as:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta y)^2}$$

Implicit methods offer great advantage on the stability of the solution, since most of them are unconditionally stable. Therefore, even choosing a larger time step, the solution will not be oscillating. On the contrary, a larger time step will decrease the accuracy of the solution, owing to the fact that the truncation error is proportional to Δt.

- Cranck-Nicholson method

If the diffusion term in eq.(1) -right hand side term- is written as the average of the central differences at time levels n and n+1, the discretized equation would be of the form:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu\left(\frac{1}{2}\right)\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta y)^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{(\Delta y)^2}\right)$$

In this way we obtain a central difference representation for the time derivative.

In fact, with some algebra, we can recognize that

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{2\left(\frac{\Delta t}{2}\right)} + O(\Delta t^2)$$

This implicit method is therefore unconditionally stable and is second-order accurated $(\Delta t^2; \Delta y^2)$.

Simulations

In this section a parabolic PDE equation will be solved by means of the different methods exposed above.
Consider a fluid bounded by two parallel plates extended to infinity, in order to neglect end-effect. The planar walls and the fluid are initially (t<0) at rest.
For $t>0$, the lower wall is instantaneously accelerated, in such a way that the plate can be considered at a given –constant- velocity.
A spatial coordinate system is selected so that the lower wall include the *xz* plane to which the *y*-axis is perpendicular; the spacing between the two plates is denoted by *h*.
I remind here that the Navier-Stokes equation for this problem may be expressed as eq (1), rewritten here for sake of completeness:

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$

(ν is the kinematic viscosity of the fluid)

The value of the simulation parameters are:

$$u_w = 40 m/s$$
$$\nu = 0.000217 m^2/s$$
$$h = 40 mm$$

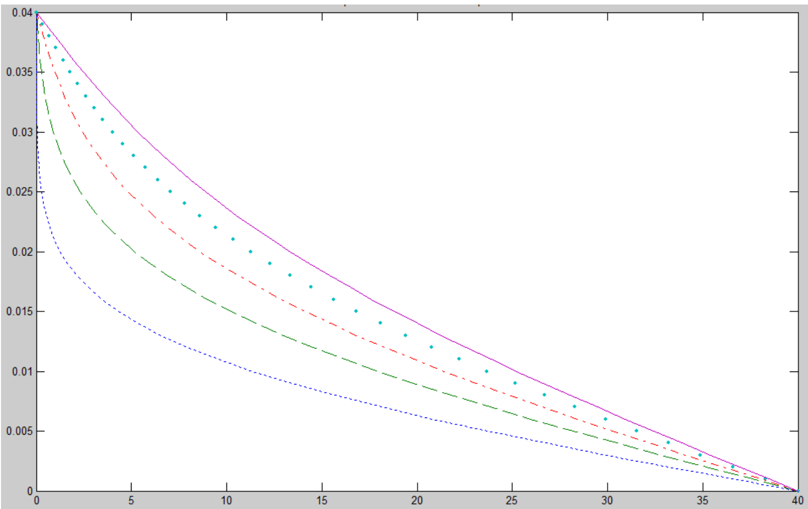
In the following, two types of boundary conditions will be used, namely Dirichlet and Neumann conditions.
A grid system with $\Delta y = 0.001$ (number of grid points, $NY=41$) and various values of time steps is used to investigate the different integration schemes and the effect of time step on stability and accuracy.
The figures shown below report the time evolution of the solution $u(y,t)$. In particular, the different curves are nothing else than the solution function taken at some time instants (the sequence of the instants is from blue -the first- to pink -the last-).

FTCS scheme (explicit)

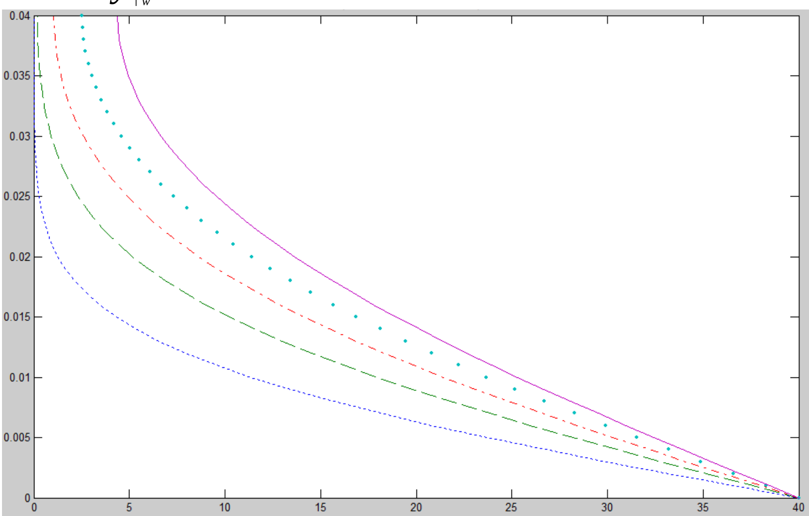
$$\Delta y = 0.001; NY = 41$$

$$\Delta t = 0.002; NT = 500$$

1) *BC at the upper wall: Dirichlet Boundary condition*
   $u_{up} = 0$



2) *BC at the upper wall: Neumann Boundary conditions*
   $\left.\frac{du}{dy}\right|_{w} = 0$

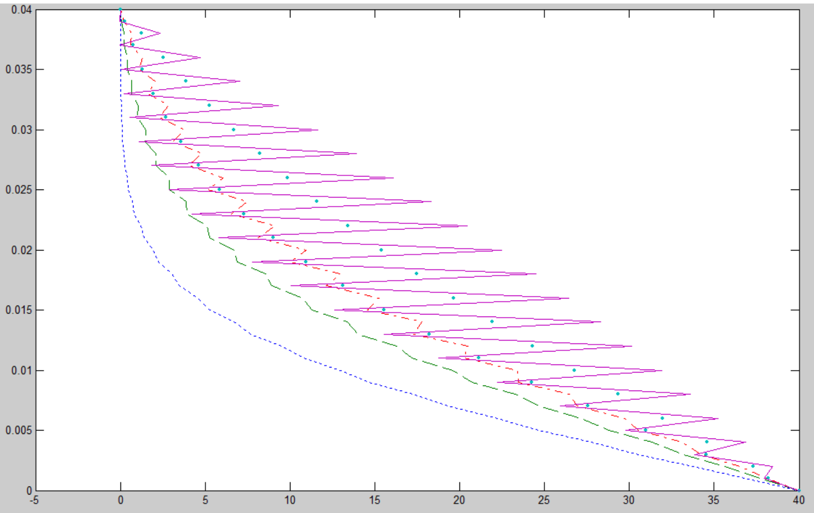


FTCS scheme (esplicit) - part 2

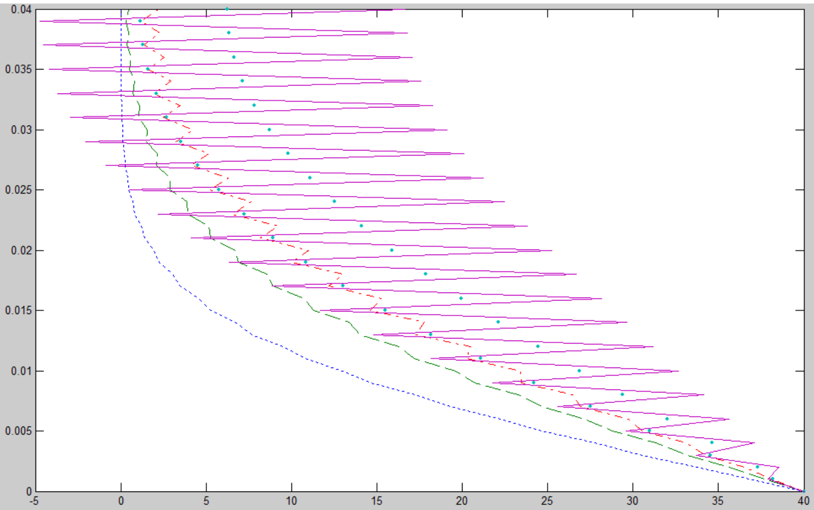
$$\Delta y = 0.001; NY = 41$$

$$\Delta t = 0.00232; NT = 500$$

1) *BC at the upper wall: Dirichlet Boundary condition*
   $u_{up} = 0$



2) *BC at the upper wall: Neumann Boundary conditions*
   $\left.\frac{du}{dy}\right|_w = 0$



Discussion:
As previously mentioned, the FTCS method is an explicit method. The stability requirement of this method is

$$\nu \frac{\Delta t}{(\Delta y)^2} \leq 0.5$$

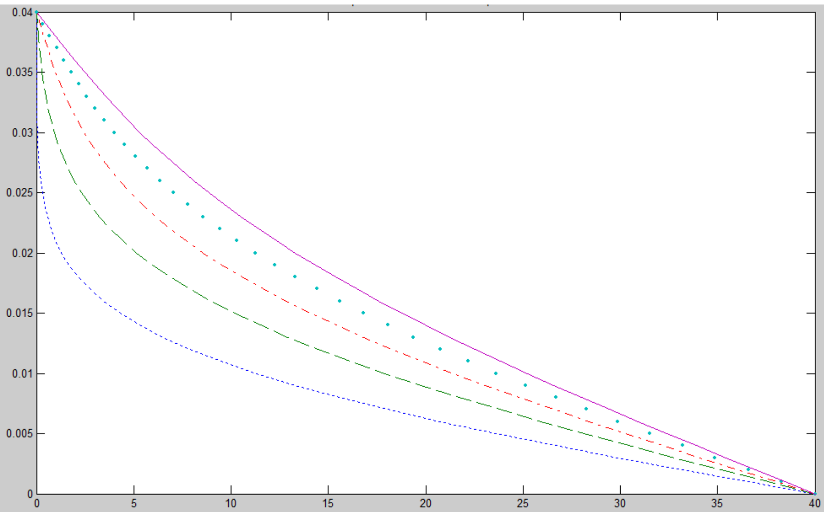
In the simulation 1), the stability requirements is satisfied (d=0.434), and a stable solution is obtained. In simulation 2), on the other hand, the diffusion number exceed the stability requirement (d=0.5034) and an unstable solution develops.

Laasonen *scheme (implicit)*

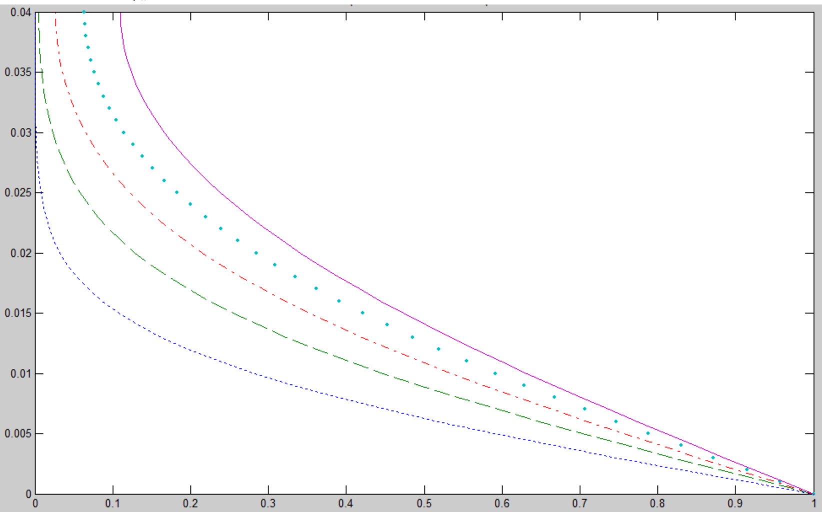
$$\Delta y = 0.001; NY = 41$$

$$\Delta t = 0.002; NT = 500$$

1) *BC at the upper wall: Dirichlet Boundary condition* $u_{up} = 0$



2) *BC at the upper wall: Neumann Boundary conditions* $\left.\frac{du}{dy}\right|_w = 0$

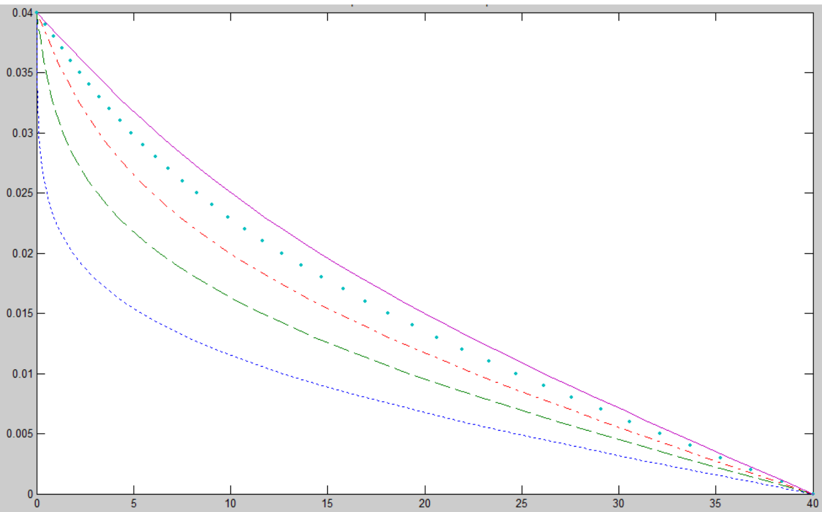


Laasonen *scheme (implicit) -part2*

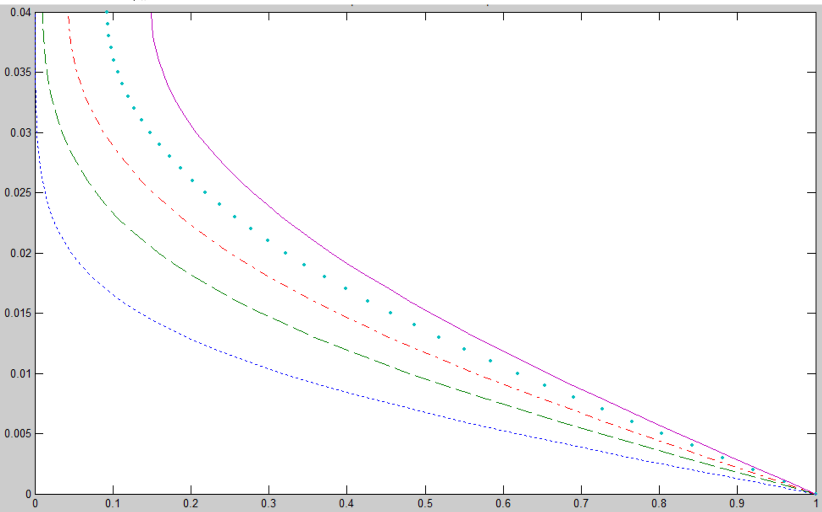
$$\Delta y = 0.001; NY = 41$$

$$\Delta t = 0.00232; NT = 500$$

1) *BC at the upper wall: Dirichlet Boundary condition*
   $u_{up} = 0$



2) *BC at the upper wall: Neumann Boundary conditions*
   $\left.\frac{du}{dy}\right|_{w} = 0$



Discussion:

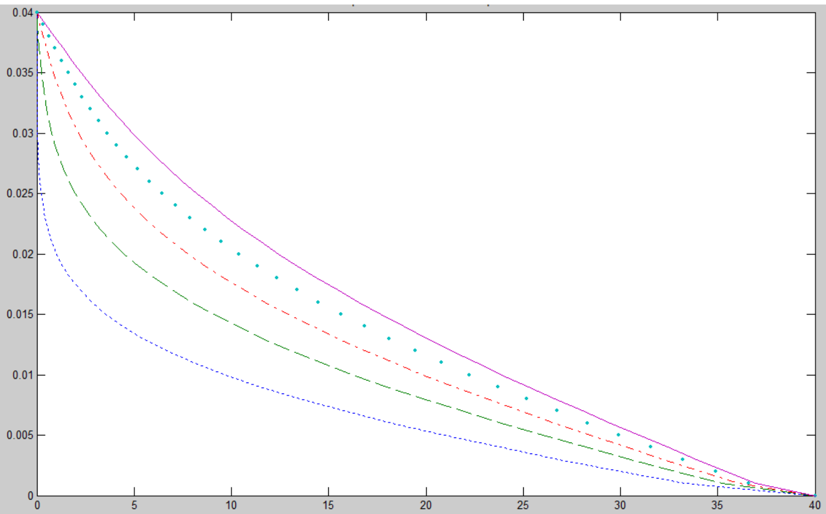
From the comparison between these results (Lasoonen scheme) and the previous ones (FTCS scheme) we can clearly observed the superiority of the implicit schemes regarding the stability of the solution of a differential equations.

<u>Cranck-Nicholson *scheme (implicit)*</u>

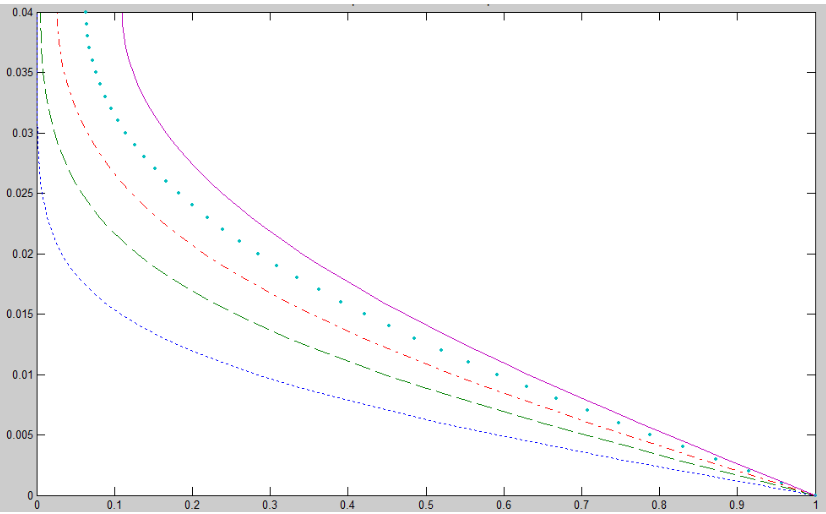
$$\Delta y = 0.001; NY = 41$$

$$\Delta t = 0.002; NT = 500$$

1) *BC at the upper wall: Dirichlet Boundary condition*
   $u_{up} = 0$



2) *BC at the upper wall: Neumann Boundary conditions*
   $$\left.\frac{du}{dy}\right|_{w} = 0$$

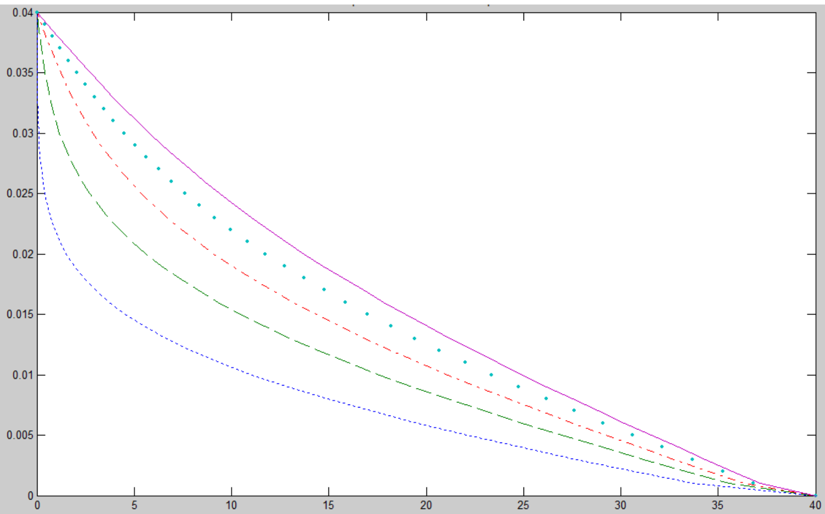


<u>Cranck-Nicholson *scheme (implicit)*</u>

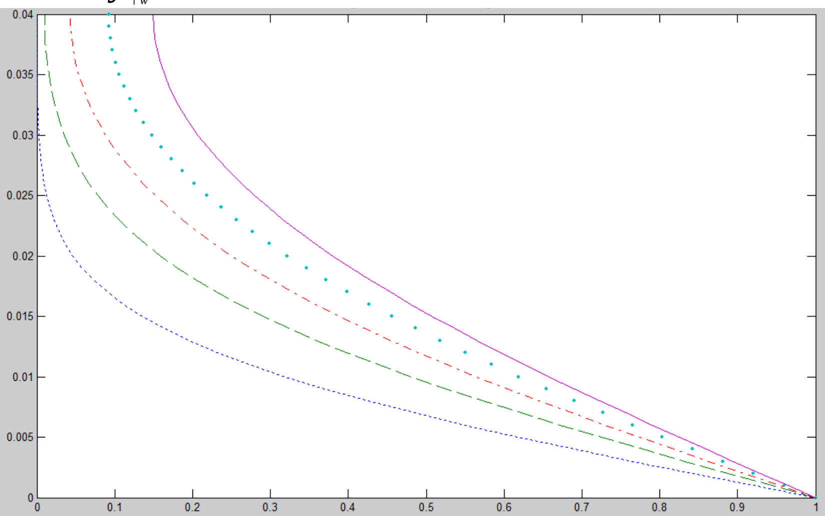
$$\Delta y = 0.001; NY = 41$$

$$\Delta t = 0.00232; NT = 500$$

1) *BC at the upper wall: Dirichlet Boundary condition*
   $u_{up} = 0$



2) *BC at the upper wall: Neumann Boundary conditions*
   $\left.\frac{du}{dy}\right|_{w} = 0$



Discussion:
As visible by the results reported above, the Cranck-Nicholson scheme is unconditionally stable (implicit scheme) and leads to a better accuracy (2^{nd} order) in comparison with the Laasonen scheme (1^{st} order). The demonstration of this topic is left to the reader (compare the various solution obtained against the "analytical" one).