Topological Defects in Fibre Suspensions

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This talk presents ideas discussed in:

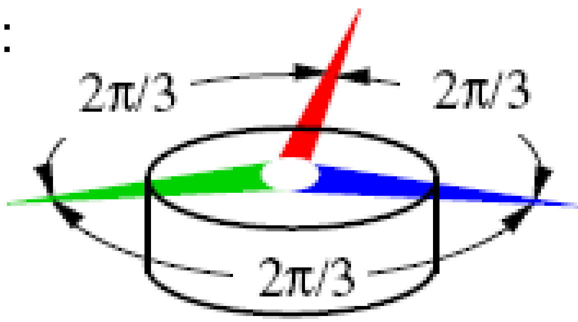
Fingerprints of Random Flows?,
M. Wilkinson, V. Bezuglyy and B. Mehlig, *Phys. Fluids*, **21**, 043304, (2009).

Poincare Indices of Rheoscopic Visualisations,
V. Bezuglyy, B. Mehlig and M. Wilkinson, *Europhys. Lett.*, **89**, 34003, (2010).

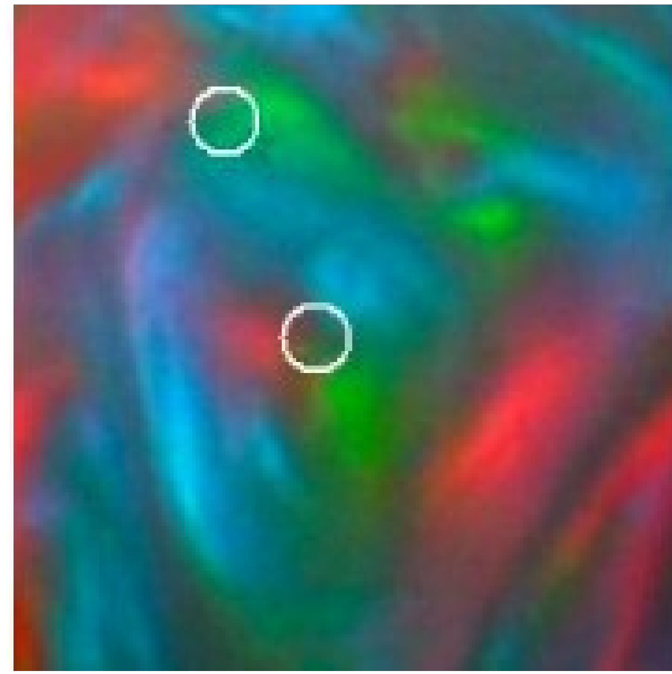
Emergent Order in Rheoscopic Swirls,
M. Wilkinson, V. Bezuglyy and B. Mehlig, *J. Fluid Mech.*, **667**, 158-87, (2011).

A simple experiment

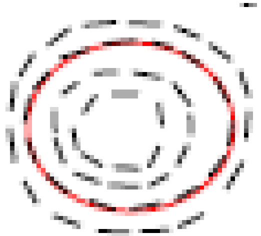
Rheoscopic fluid is a suspension of rod-like crystals, used for flow visualisation. This is a photograph of randomly-stirred fluid in a shallow dish:



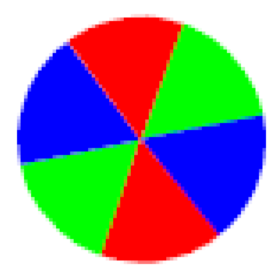
S. T. Thoroddsen and J. M. Bauer, *Qualitative flow visualization using colored lights and reflective flakes*, *Phys. Fluids*, 11, 1702, (1999).



There are singularities about which the colours are encountered in cyclic order. Are these vortices?

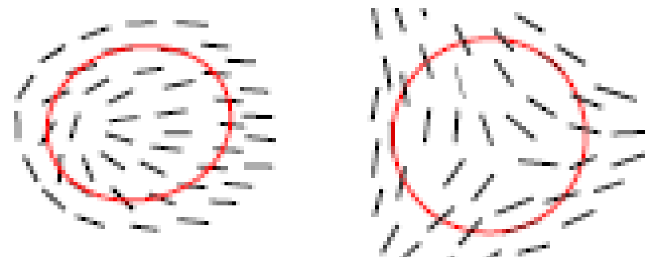


No, because the colours would cycle twice:

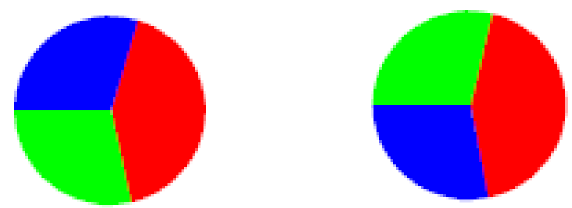


Interpretation of the singularities

The rod-like crystals define a non-oriented vector field (reversing the vector has no significance). Another example is the ridge pattern of fingerprints. These do have point singularities:



core *delta*



Poincare index N is defined for of a curve lying in a vector field. It is number of multiples of 2π by which vector rotates (clockwise) on traversing a closed curve (also clockwise). A curve with non-zero index encloses a singularity. Experiment indicates that there are singularities with Poincare index $\pm\frac{1}{2}$. The core has index $\frac{1}{2}$, the delta has index $-\frac{1}{2}$.

Exact solution of the equation of motion

Equation of motion for rods depends upon velocity gradient of fluid:

$$\dot{\mathbf{n}} = \mathbf{A}\mathbf{n} - (\mathbf{n}\cdot\mathbf{A}\mathbf{n})\mathbf{n} \qquad A_{ij} = \partial v_i / \partial r_j$$

G. B. Jeffery,
The motion of ellipsoidal particles immersed in a viscous fluid,
Proc. R. Soc. London, Ser. A, **102**, 161, (1922).

Equation may be solved using a solution of a companion linear equation:

$$\dot{\boldsymbol{d}} = \mathbf{A}(t)\boldsymbol{d} \qquad \mathbf{n}(t) = \frac{\boldsymbol{d}(t)}{|\boldsymbol{d}(t)|}$$

A. J. Szeri,
Pattern formation in recirculating flows of suspensions of orientable particles,
Phil. Trans. R. Soc. Lond., **A345**, 477-508, (1993).

For general initial condition, we should compute the monodromy matrix:

$$\boldsymbol{d}(t) = \mathbf{M}(t, t_0)\mathbf{n}_0 \qquad \dot{\mathbf{M}} = \mathbf{A}(\boldsymbol{r}(t), t)\mathbf{M} \qquad \delta\boldsymbol{r}(t) = \mathbf{M}(t, t_0)\delta\boldsymbol{r}(t_0)$$

Are singularities forbidden?

General solution, in terms of the initial field, $\mathbf{n}_0(\boldsymbol{r})$:

$$\mathbf{n}(\boldsymbol{r},t) = \frac{\mathbf{M}(\boldsymbol{r},t,t_0)\mathbf{n}_0(\boldsymbol{r}_0)}{|\mathbf{M}(\boldsymbol{r},t,t_0)\mathbf{n}_0(\boldsymbol{r}_0)|}$$

Monodromy matrix is non-singular: $\det(\mathbf{M}) = 1$. This implies that the solution is non-singular.

However, if the initial orientation is random, we should average over the initial direction $\mathbf{n}_0$, assumed uniformly distributed around a unit circle.

The unit circle is mapped by the matrix $\mathbf{M}$ to an ellipse with aspect ratio ν , with long axis in direction $\bar{\theta}$. This ellipse is:

$$\boldsymbol{x}\cdot\mathbf{K}\boldsymbol{x} = 1 \ , \quad \mathbf{K} = (\mathbf{M}\mathbf{M}^{\mathrm{T}})^{-1}$$

Definition of an order parameter

The distribution of directions is:

$$P(\theta) = \frac{\nu}{2\pi} \frac{1}{(\nu^2 - 1)\sin^2(\theta - \bar{\theta}) + 1}$$

Distribution of angles is characterised by an order parameter:

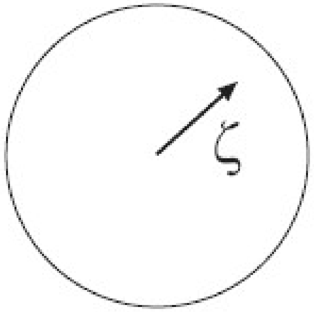
$$\boldsymbol{\zeta} = \frac{\nu - 1}{\nu + 1}\, \mathbf{n}(\bar{\theta})$$

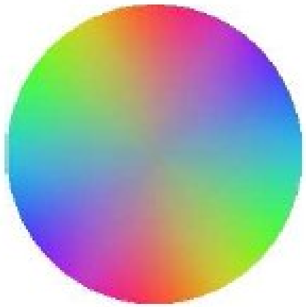
(perfect alignment when $|\zeta| = 1$, random orientations give $\boldsymbol{\zeta} = \mathbf{0}$).

The order parameter can be related to the colour of scattered light:

$$C = I(0)\, R + I(2\pi/3)\, G + I(4\pi/3)\, B$$

$$I(\phi) = \int_0^{2\pi} \mathrm{d}\theta\ P(\theta) \cos^2(\phi - \theta)$$





Zeros of the order parameter

Order parameter is zero when the distribution of directions is uniform. This happens when the monodromy matrix $\mathbf{M}$ is a rotation matrix $\mathbf{O}(\chi)$. General parametrisation:

$$\mathbf{M} = \mathbf{D}(\lambda, \lambda^{-1})\,\mathbf{S}(\kappa)\,\mathbf{O}(\chi)$$

$$\mathbf{D}(\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \;, \quad \mathbf{S}(\kappa) = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{O}(\chi) = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \;.$$

Zeros occur upon varying two parameters, so that $\lambda = 1$, $\kappa = 0$.

In vicinity of a zero at $\boldsymbol{r}_0$, we can use coordinates (X, Y) such that

$$\mathbf{M}(\boldsymbol{R}) = \mathbf{D}(1 + \tfrac{1}{2}X, 1 - \tfrac{1}{2}X)\,\mathbf{S}(\pm Y)\,\mathbf{O}(\chi) + \mathcal{O}(\boldsymbol{R}^2)$$

$$\boldsymbol{r} - \boldsymbol{r}_0 = \mathbf{T}\boldsymbol{R}$$

$$\boldsymbol{R} = (X, Y)$$

Are these normal forms for zeros ‘fingerprint’ singularities?

Structure of zeros of the order parameter

Normal form: $$\mathbf{M}(\boldsymbol{R}) = \mathbf{D}(1+\tfrac{1}{2}X, 1-\tfrac{1}{2}X)\,\mathbf{S}(\pm Y)\,\mathbf{O}(\chi) + \mathcal{O}(\boldsymbol{R}^2)$$

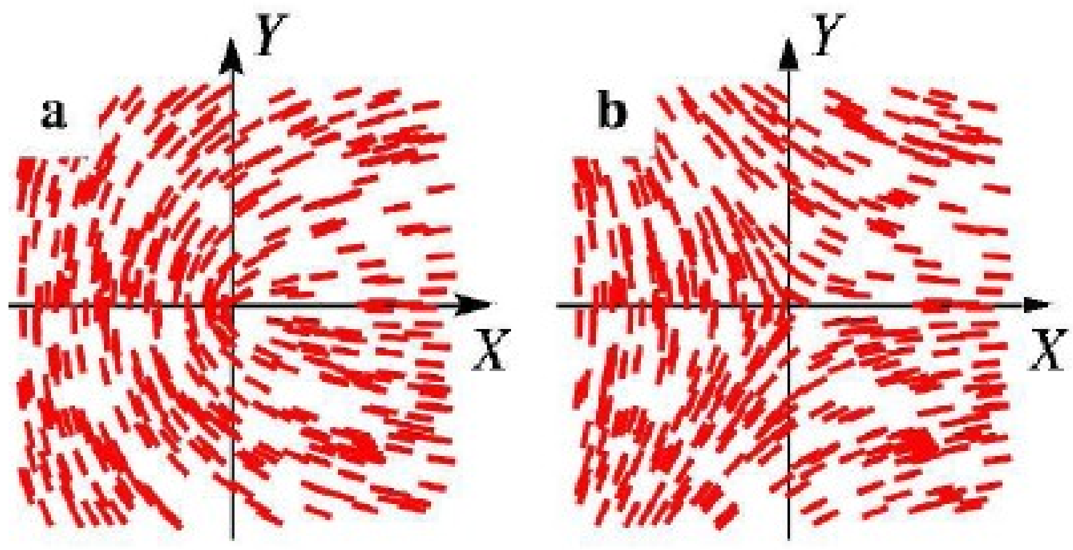
The matrix of the quadratic form describing the ellipse is:

$$\mathrm{M}\,\mathrm{M}^{\mathrm{T}} = \begin{pmatrix} 1+X & \pm Y \\ \pm Y & 1-X \end{pmatrix} + \mathcal{O}(R^2)$$

The order parameter is:

$$\boldsymbol{\zeta}(R,\Theta) = R\,\mathbf{n}(\pm\tfrac{1}{2}\Theta) + \mathcal{O}(R^2) \qquad (X,Y) = (R\cos\Theta, R\sin\Theta)$$

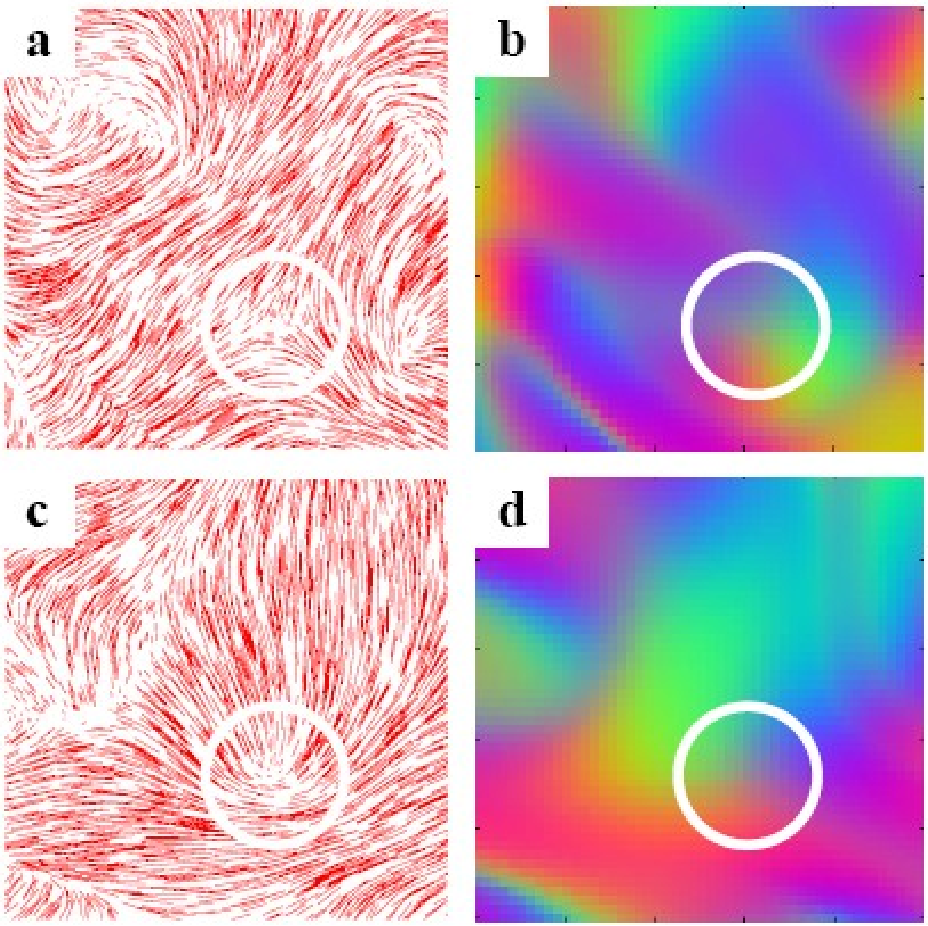
core



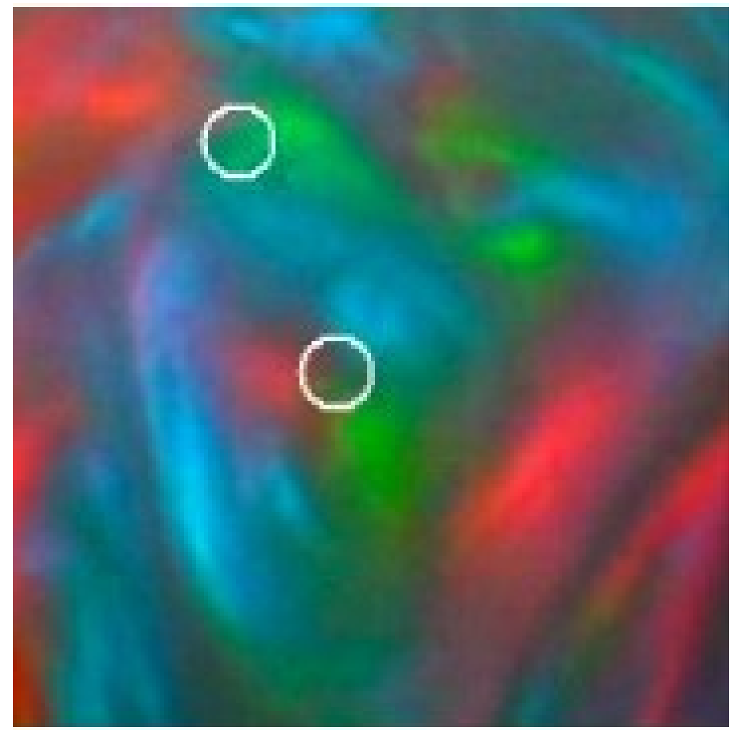
delta

Comparing experiment and simulation

Simulation (using a random flow field):



Experiment:

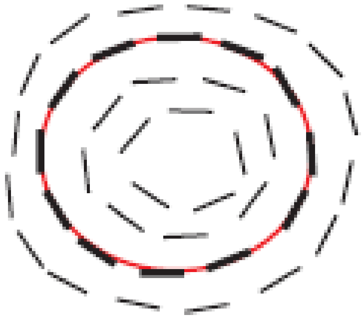


The case of steady flows

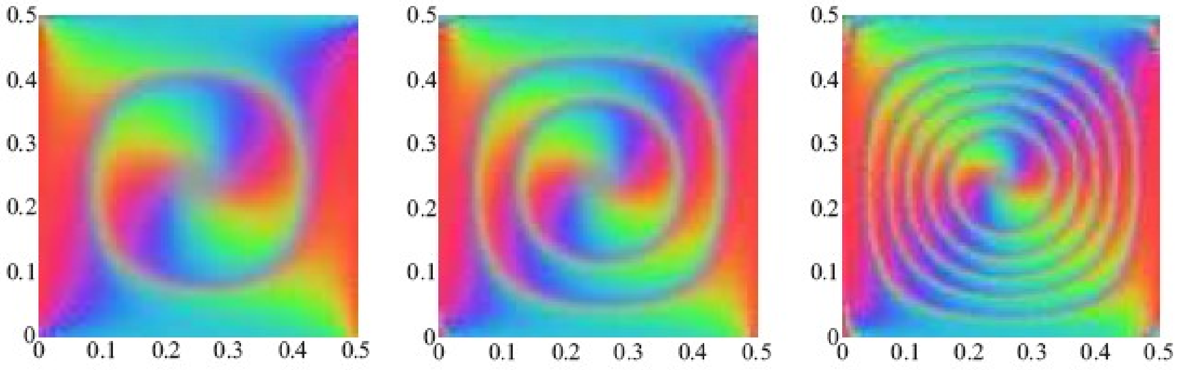
Steady two-dimensional flows are described by a stream function, $\psi(x,y)$

$$\dot{x} = \frac{\partial \psi}{\partial y}\,, \quad \dot{y} = -\frac{\partial \psi}{\partial x}$$

Particles follow the contours of the stream function. The monodromy matrix is a shear, and rod-like particles align with contours of the stream function:

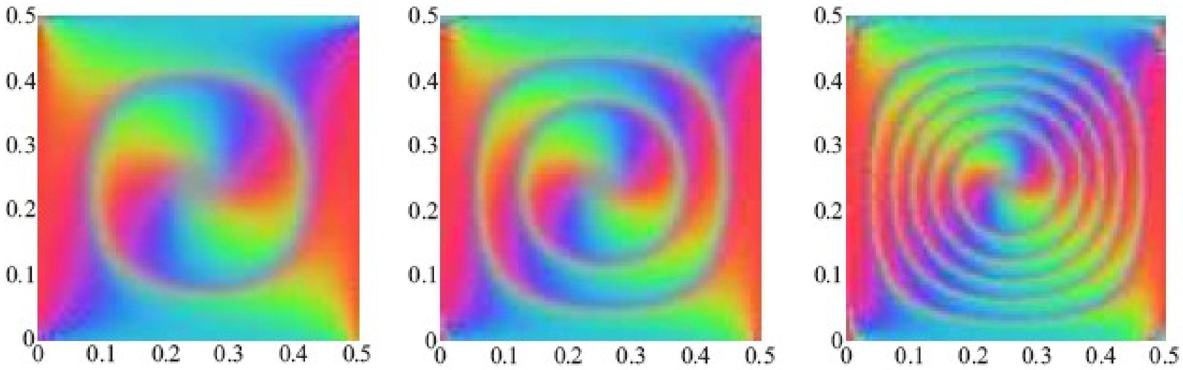


Particles with finite aspect ratio tumble in a shear flow. The order parameter exhibits an increasingly tight spiral pattern:



The long-time limit

In the elliptic bands the order parameter forms a tightening spiral:



Local average $\langle \zeta \rangle$ converges in long-time limit. After an involved argument, we find that the averaged order parameter is determined by the normal form of the transfer matrix:

$$\mathbf{M}_0 = \mathbf{X}\,\mathbf{R}(\theta_0)\,\mathbf{X}^{-1}$$

Just replace $\mathbf{M}$ by $\mathbf{X}$ in earlier formulae for the order parameter. It follows that the singularities of $\langle \zeta \rangle$ are cores and deltas.

Elliptic and Hyperbolic bands

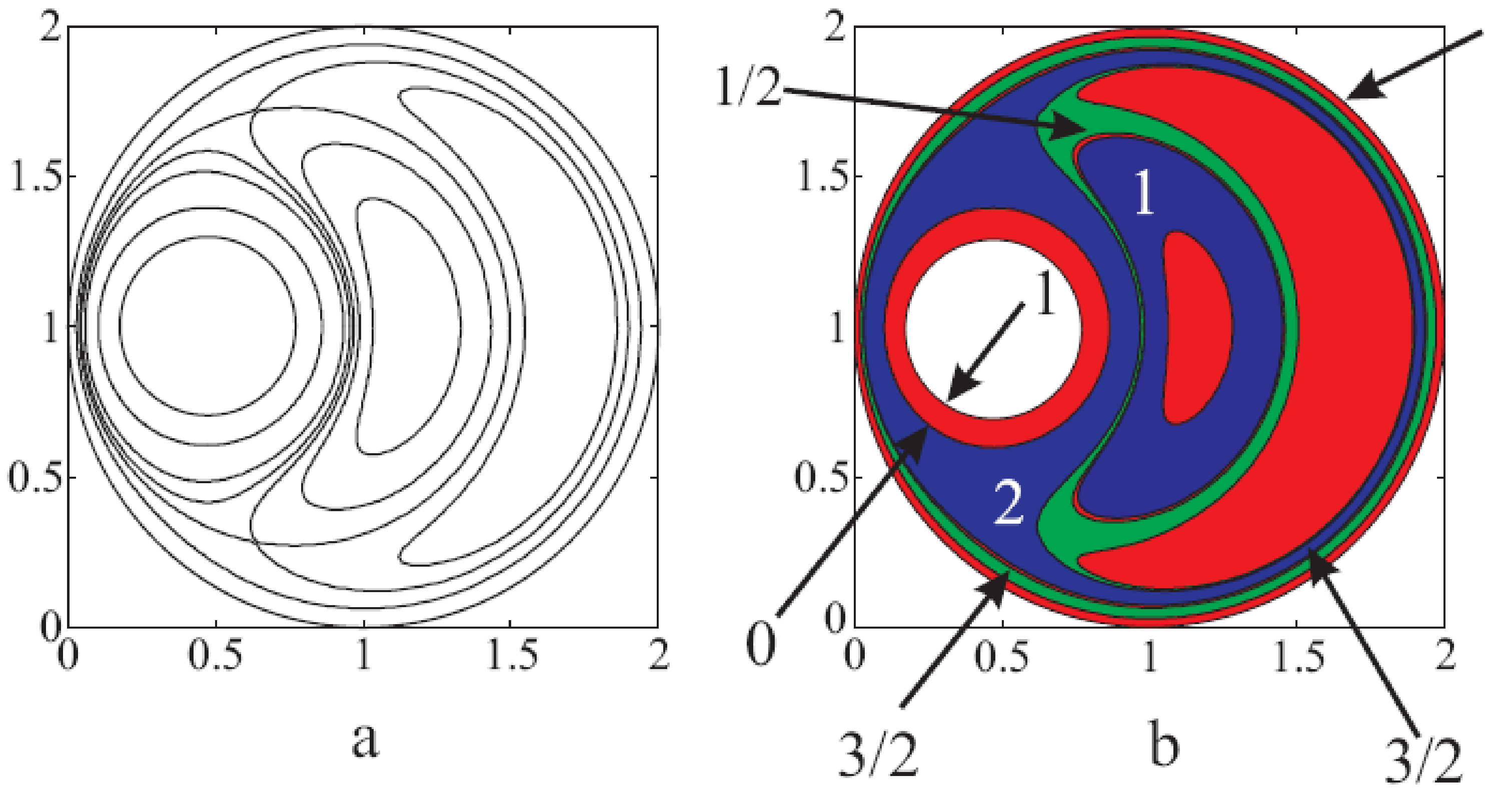


FIG. 6: a Contours of the stream function $\psi(x, y)$ for a journal bearing system: both circular boundaries rotate in the same direction, with the angular speed of the inner boundary exceeding that of the outer boundary by a factor of 20. b These contours can be coloured according to whether the transfer matrix M_0 is elliptic $|\mathrm{tr}(M_0)| < 2$ (red), or hyperbolic, $\mathrm{tr}(M_0) > 2$ (blue) and $\mathrm{tr}(M_0) < -2$ (green). Each hyperbolic band is labelled by its Poincaré index. In this illustration we set $\alpha_1 = 0.95$, $\alpha_2 = 0.05$ in equations (1), (2), (which corresponds to an ellipse with aspect ratio $\beta = \sqrt{19} = 4.36..$).

Averaged order parameter in long-time limit

Note that $\langle \zeta \rangle$ is not reflection symmetric, but its zeros are symmetric.

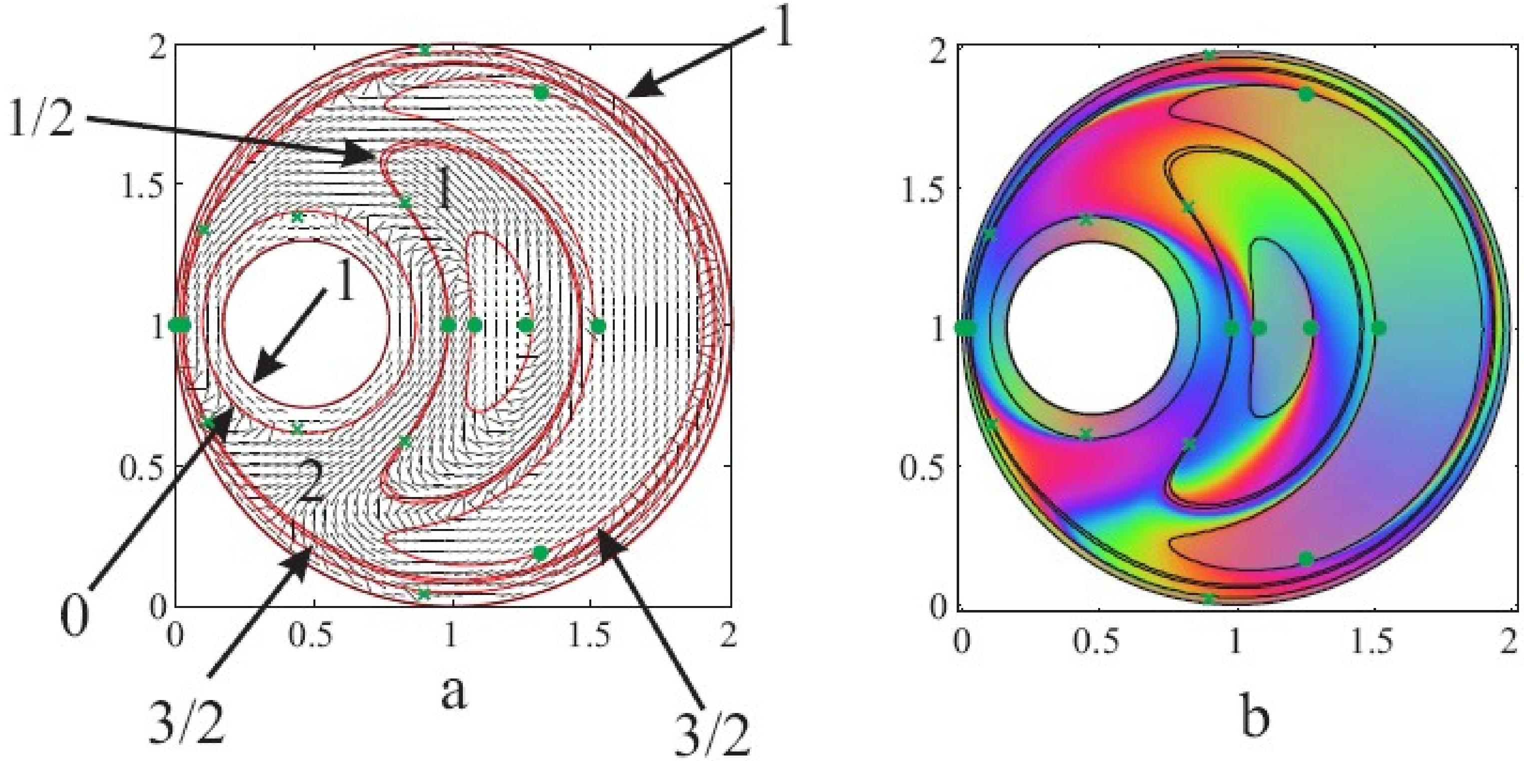
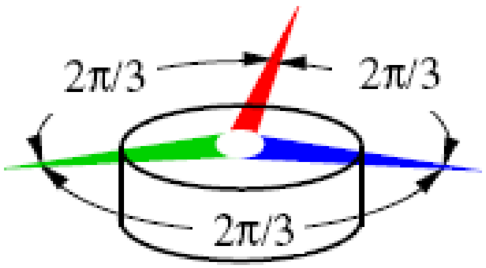
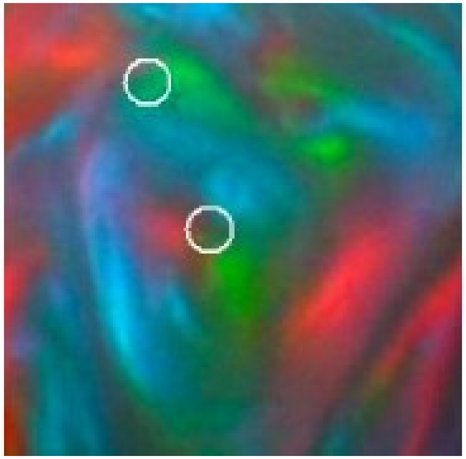


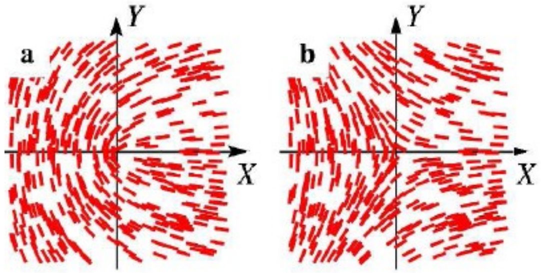
FIG. 10: a Illustrating the locally averaged order parameter field computed using (23) and (26). The hyperbolic and elliptic bands are separated by red lines, and the positions of zeros of the order parameter field are indicated by green dots for zeros with Poincaré index equal to $\frac{1}{2}$, green crosses for zeros with index $-\frac{1}{2}$. b Is the same image as figure 4, with the boundaries between hyperbolic and elliptic bands indicated by solid black lines.

Summary and Open Question

Experiment shows the existence of fingerprint-like singularities in alignment of rod-like objects. We have analysed their normal forms.







These topological singularities must *influence* and perhaps *organise* the rheological behaviour of fibre suspensions: how can this be investigated and quantified?