A critical examination of the Basset-Boussinesq force

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The Basset-Boussinesq force

The force on a steadily-moving sphere was obtained by Stokes (1851). Non-steady motion was treated by Boussinesq (1885) and Basset (1888). A term depending upon the history of the motion has a slow ($t^{-1/2}$) decay of the response kernel:

$$F(t) = \frac{2\pi}{3}\rho a^3 \ddot{X}(t) + 6\pi\rho\nu a \dot{X}(t)$$

$$+6a^2\sqrt{\pi\nu}\int_{-\infty}^{t} \mathrm{d}t' \frac{\ddot{X}(t')}{\sqrt{t-t'}}$$

Recent works:

M. R. Maxey and J. J. Riley, *Phys. Fluids* **26**, 883 (1983)

R. Gatignol, *J. Mec. Theor. Appl.*, **1**, 143 (1983).

R. Mei and R. J. Adrian, *J. Fluid. Mech.*, **237**, 323-41, (1992).

M. Parmar, A. Haselbacher, and S. Balachandar, *Phys. Rev. Lett.*, **106**, 084501, (2011).

A. Daitche and T. Tl, *Phys. Rev. Lett.*, **107**, 244501, (2011).

Diffusion of vorticity

The form of the Boussinesq-Basset force suggests that it is related to a diffusion process. At low Reynolds numbers the vorticity satisfies a diffusion equation.

$$\boldsymbol{\Omega} = \boldsymbol{\nabla} \wedge \boldsymbol{u}$$

$$\frac{D\boldsymbol{\Omega}}{Dt} \equiv \frac{\partial \boldsymbol{\Omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\Omega} = \mathbf{A}\boldsymbol{\Omega} + \nu \boldsymbol{\nabla}^2 \boldsymbol{\Omega}$$

Solution may be obtained by integrating over a propagator e.g. in one dimension

$$\Omega(y,t) = \int_{-\infty}^{t} \mathrm{d}t' \; J(t') G(y, t - t')$$

$$G(y,t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left(-\frac{y^2}{4\nu t}\right)$$

Force on a plate

To understand the origin of the history force, consider a simpler problem: the shear stress on an infinite plate moving tangentially. This is found to be

$$\sigma(t) = \rho\sqrt{\frac{\nu}{\pi}}\int_{-\infty}^{t} \mathrm{d}t' \frac{\ddot{X}(t')}{\sqrt{t-t'}}$$

To interpret this result: note that .

$$\Omega(y,t) = \frac{\partial u_x}{\partial y}(y,t) \qquad \dot{X}(t) = u_x(0,t) = \int_0^\infty \mathrm{d}y\ \Omega(y,t)$$

$$\sigma(t) = \nu\rho\Omega(0,t) \qquad J(t) = \ddot{X}(t)$$

and use the one-dimensional diffusive propagator. .

What is different about a sphere?

Motion of the sphere creates vorticity, which diffuses away from its surface. At short times this is similar to the moving plate problem. At longer times, satisfying

$$\nu t \gg a^2$$

the three-dimensional spherically symmetric diffusion propagator is expected to be relevant:

$$\Omega(r,t) = t^{-3/2} \exp\left(-\frac{r^2}{4\nu t}\right)$$

The kernel of the history force has two asymptotes:

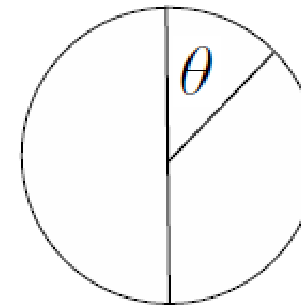
Short time: $t^{-1/2}$

Long time: $t^{-3/2}$

Vorticity diffusion from a sphere

Assume vorticity distribution is dipolar:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nu \nabla^2 \boldsymbol{\Omega} \qquad \boldsymbol{\Omega} = f(r,t)\,\sin\theta\,\mathbf{e}_\phi$$



Close to the sphere, the distribution of vorticity resembles a uniformly moving sphere: the Stokes solution gives

$$f(r) = \frac{3Ua}{2r^2}$$

For unsteady flow there is a time-dependent source of vorticity, proportional to the applied force

$$J(t) = \frac{1}{2\pi a^3 \rho} F(t) \qquad f(r,t) \sim \frac{J(t)a^2}{2\nu r^2}$$

Relating vorticity and velocity

Diffusion propagator for dipolar source:

$$\frac{\partial f}{\partial t} = \nu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) - \frac{l(l+1)}{r^2} f \right] \qquad l = 1$$

$$f(r,t) = A t^{-5/2} r \exp\left(-\frac{r^2}{4\nu t} \right)$$

Kinematic relation between velocity and vorticity:

$$u_r(r,\theta,t) = \cos\theta v_r(r,t)$$

$$v_r(r,t) = \int_a^r \mathrm{d}r' \, W(r,r') \, f(r',t)$$

$$W(r,R) = \frac{2}{3} R^3 \left(\frac{1}{r^3} - \frac{1}{R^3} \right)$$

Long-time propagator

Relate particle velocity to vorticity field

$$\dot{X}(t) = -\lim_{r\to\infty} v_r(r,t)$$

$$v_r(r,t) = \int_a^r \mathrm{d}r'\; W(r,r')\, f(r',t)$$

$$f(r,t) = \int_{-\infty}^t \mathrm{d}t'\; P(r,t-t')\, J(t')$$

$$J(t) = \frac{1}{2\pi a^3 \rho} F(t)$$

Combine these to obtain

$$\dot{X}(t) = \int_{-\infty}^t \mathrm{d}t'\; \Gamma(t-t')\, F(t')$$

The result

The velocity is expressed as a history integral over the force:

$$\dot{X}(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \Gamma(t - t') \, F(t')$$

The kernel is

$$\Gamma(t) = \frac{1}{6\pi\rho\nu} \frac{1}{\sqrt{4\pi\nu}} t^{-3/2} \exp\left(-\frac{a^2}{4\nu t}\right)$$

Response to a steady force

$$\dot{X} = F \int_{-\infty}^{0} \mathrm{d}t \; \Gamma(t) = \frac{F}{6\pi\rho\nu a}$$

Conclusion

The kernel in the Basset-Boussinesq expression for the history force is incorrect at long times. The correct long-time asymptote has been determined for the velocity expressed in terms of the force:

$$\dot{X}(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \Gamma(t-t') \, F(t')$$

$$\Gamma(t) = \frac{1}{6\pi\rho\nu} \frac{1}{\sqrt{4\pi\nu}} t^{-3/2} \exp\left(-\frac{a^2}{4\nu t}\right)$$

There are difficulties in observing the Basset-Boussinesq force experimentally. Mei and Adrian proposed a semi-empirical kernel with t^{-2} decay:

R. Mei and R. J. Adrian, *J. Fluid. Mech.*, **237**, 323-41, (1992).