The fluid velocity field is therefore

$$v_x = V\frac{y}{h}. \tag{2.7.10}$$

The shear stress acting on the plate is

$$\tau_{yx} = \mu\frac{dv_x}{dy}\bigg|_{y=h} = \mu\frac{V}{h}. \tag{2.7.11}$$

The shear stress acts in the positive x direction. The plate creates a shear stress that causes the fluid to move. Thus, the direction of the shear stress and the fluid motion are the same in this case.

Example 2.6 Two parallel plates are separated by a thin gap of thickness h containing a Bingham plastic, as shown in Figure 2.22a. A force per unit area, τ_{yx}, is exerted on the upper plate. Under what conditions does the upper plate move?

Solution Equation (2.7.5) is valid for a fluid moving between parallel plates. Equation (2.7.7) indicates that the shear stress is uniform throughout the fluid. The upper plate moves when $\tau_{yx} > \tau_0$. For $\tau_{yx} < \tau_0$, v_x is constant. The lower plate is not moving, so $v_x\ (y=0)=0$. Since v_x is a constant everywhere between the two plates and v_x must equal zero at $y=0$, v_x must equal zero at all points in the range $0 < y < h$. Therefore, the upper plate moves only when the yield stress is exceeded. The velocity field is

$$\tau_{yx} < \tau_0 \qquad v_x = 0, \tag{2.7.12a}$$

$$\tau_{yx} > \tau_0 \qquad v_x = \frac{(\tau_{yx} - \tau_0)y}{\mu_0}. \tag{2.7.12b}$$

Note that, for a Bingham plastic, the velocity of the upper plate is less than the velocity that would exist for a Newtonian fluid (i.e., $\tau_0 = 0$).

2.7.2 Pressure-Driven Flow through a Narrow Rectangular Channel

The next two cases characterize flow through channels. Flow is induced by a pressure gradient, which could be produced by a pump or gravity. These examples also examine the effect of geometry upon the flow and shear stress. In the first example, we examine flow through a narrow rectangular channel of height h and width w (Figure 2.23). Flow through this kind of channel is important in many biomedical devices, such as hemodialyzers and ultrafiltration units. Parallel-plate channels are used widely to study the effect of flow on cell adhesion and cell function.

Poiseuille

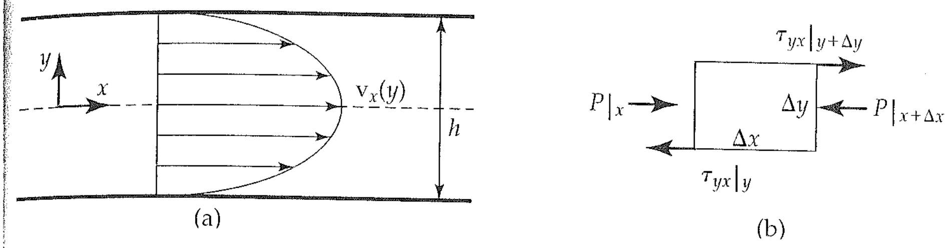

FIGURE 2.23 (a) Pressure-driven flow through a rectangular channel. Note that the velocity profile is symmetric around the centerline. The channel thickness is h and the channel width is w. (b) Momentum balance applied to a cubic control volume.

The velocity profile is sketched in Figure 2.23a. Due to the no-slip boundary condition, the velocity is zero at either surface ($y = h/2$). In order to analyze this flow, we must make the following assumptions:

1. The pressure varies only in the direction of flow.
2. The fluid density is constant, which indicates that the fluid is incompressible.
3. The flow is steady; that is, pressure, shear stress, and velocity do not change with time.
4. The fluid is Newtonian.
5. Flow is fully developed. This means that the channel length is much longer than the entrance length L_e where the velocity depends upon axial distance along the channel, $L_e \ll L$.
6. Edge effects are neglected. To meet this assumption, we require a long, wide channel; that is, $h/w \ll 1$ and $h/L \ll 1$, where w is the width and L is the length of the channel.
7. The flow is laminar.

The first and fifth assumptions imply that there is only one shear stress, namely, τ_{yx}. The pressure is balanced by shear stress acting in the x direction. With only one shear stress acting on the control volume, there is only one velocity component, v_x, and the flow is considered *fully developed*. This means that the shear stresses and velocity field do not change along the x direction, so v_x is a function of y only. Thus, the net flow of momentum in the x direction is zero.

Because the plates are not moving, the velocity of the fluid in contact with the upper and lower plates is zero. Further, the velocity and shear stress must be symmetric about the midplane. For this reason, the coordinate system in the y direction has its origin along the midplane between the plates. Other possible choices for the origin in the y direction include the upper and lower surfaces of the channel. The proposed velocity profile is consistent with the expected symmetry about the midplane and no-slip boundary conditions.

A control volume for analysis is shown in Figure 2.23b. Since there is no net momentum flow and the flow is steady, the sum of all forces must equal zero. The only forces arising are those due to pressure and shear stress. A momentum balance in the x direction yields

$$\left(p|_x - p|_{x+\Delta x}\right)\Delta y \Delta z + \left(\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_y\right)\Delta x \Delta z = 0. \tag{2.7.13}$$

Dividing by $\Delta x\ \Delta y\ \Delta z$ and taking the limit as Δx, Δy, and Δz each go to zero results in the following ordinary differential equation:

$$\frac{dp}{dx} = \frac{d\tau_{yx}}{dy}. \tag{2.7.14}$$

The pressure changes only in the x direction (i.e., $p = f(x)$), and the shear stress changes in the y direction ($\tau_{yx} = g(y)$). Thus,

$$\frac{df(x)}{dx} = \frac{dg(y)}{dy}. \tag{2.7.15}$$

The left- and right-hand sides of Equation (2.7.15) can be equal only if *the derivatives* are each equal to a constant C_1. Hence, the left-hand side of Equation (2.7.14) can be integrated to yield

$$p = C_1 x + C_2. \tag{2.7.16}$$

The pressure can be specified at two locations, away from both the entrance and the exit. Thus, at $x = x_0$, $p = p_0$, and at $x = x_L$, $p = p_L$. Defining $\Delta p = p_0 - p_L$ and $x_L - x_0 = L$, we find that the pressure is

$$p = p_0 + \frac{\Delta p}{L}(x_0 - x), \tag{2.7.17}$$

and Equation (2.7.14) becomes

$$-\frac{\Delta p}{L} = \frac{d\tau_{yx}}{dy}. \tag{2.7.18}$$

Integrating Equation (2.7.18) results in

$$\tau_{yx} = -\frac{\Delta p}{L}y + C_3. \tag{2.7.19}$$

The stresses are not specified on any boundary, but the velocity is known. To find the constant of integration, determine the velocity and apply the boundary condition that $v_x = 0$ at $y = \pm h/2$. To find the velocity, insert Newton's law of viscosity, Equation (2.5.6), into Equation (2.7.19) to yield

$$\mu\frac{dv_x}{dy} = -\frac{\Delta p}{L}y + C_3. \tag{2.7.20}$$

After integrating Equation (2.7.20), we have

$$v_x = -\frac{\Delta p}{2\mu L}y^2 + \frac{C_3}{\mu}y + C_4. \tag{2.7.21}$$

Applying the boundary conditions results in $C_3 = 0$ and $C_4 = \Delta p h^2/8\mu L$. The velocity profile is

$$v_x = \frac{\Delta p h^2}{8\mu L}\left(1 - \frac{4y^2}{h^2}\right). \tag{2.7.22}$$

Note that v_x is symmetric around $y = 0$, which indicates that the velocity gradient and shear stress are zero at $y = 0$. The *symmetry condition* that the velocity gradient is zero along the centerline is used in subsequent examples and problems.

Before examining the stress, we need to compute several results with respect to the velocity and flow rate. Equation (2.7.22) describes a parabola and the velocity is a maximum at $y = 0$ with a value of

$$v_{\max} = \frac{\Delta p h^2}{8\mu L}, \tag{2.7.23}$$

and Equation (2.7.22) can be written as

$$v_x = v_{\max}\left(1 - \frac{4y^2}{h^2}\right). \tag{2.7.24}$$

The volumetric flow rate is the integral of the velocity over the cross-sectional area through which fluid flows; that is,

$$Q = \int_{-h/2}^{h/2}\int_0^w v_x dz dy = v_{\max} w \int_{-h/2}^{h/2}\left(1 - \frac{4y^2}{h^2}\right)dy \tag{2.7.25}$$

or

$$Q = v_{\max} w\left(y - \frac{4y^3}{3h^2}\right)\Bigg|_{-h/2}^{h/2} = \frac{2v_{\max} wh}{3}. \tag{2.7.26}$$

The average velocity through the cross-sectional area A is

$$\langle v\rangle = \frac{1}{A}\int_A v_x dA = \frac{1}{wh}\int_{-h/2}^{h/2}\int_0^w v_x dz dy, \tag{2.7.27}$$

The volumetric flow rate can be written as

$$Q = \langle v\rangle wh. \tag{2.7.28}$$

To relate the average and maximum velocities for flow through a channel, equate Equations (2.7.26) and (2.7.28)

$$\langle v\rangle = \tfrac{2}{3}v_{\max}. \tag{2.7.29}$$

Since $C_3 = 0$, the shear stress, Equation (2.7.19), is

$$\tau_{yx} = -\frac{\Delta p}{L}y. \tag{2.7.30}$$

The shear stress distribution τ_{yx} is sketched in Figure 2.24. The shear stress is positive for $y < 0$ and negative for $y > 0$. This change in sign is due to the convention for the sign of the shear stress and to the location of the origin of the coordinate

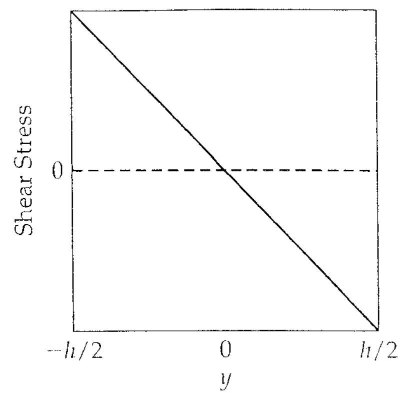

FIGURE 2.24 Shear stress distribution for flow between parallel plates.

system. For $y < 0$, a positive shear stress points in the negative x direction. For $y > 0$, the normal is in the positive y direction, and the negative sign indicates that the shear stress acts in the negative x direction.