# Chapter 1

# Mechanisms and Models of River Mixing

## 1.1 INTRODUCTION

When a tracer is introduced into a river we notice that two things happen to it. Firstly, it is carried away from the point of discharge by the current, a process termed advection. Secondly, it spreads out, a process which is termed either diffusion or dispersion depending on which mechanisms are operating. This chapter describes the physics and mathematics of advection, molecular diffusion, turbulent diffusion and dispersion. The main objectives are to help the reader understand these processes and to derive the important mixing equations. The derivation of several equations is given in detail to emphasise the roles of time and space averaging. This should help the reader understand the physical significance of each of the terms and coefficients in the mixing equations and reduce the risk of selecting inappropriate models or coefficients when tackling practical problems.

Chapter 1 is principally concerned with the theoretical basis of the mixing equations and the practicalities of solving mixing problems are discussed in Chapters 2, 3 and 4. Some readers may prefer to become familiar with the practical aspects of river mixing before returning to this more theoretical chapter. Others may want to explore the physics and mathematics of river mixing before considering the practical applications. A summary of the key points in Chapter 1 is given at the outset to help the reader see where the chapter is heading and to serve as a subsequent checklist.

## 1.2 SUMMARY

1. The two fundamental processes involved in tracer mixing are advection and molecular diffusion. Molecular diffusion can be modelled satisfactorily using a gradient diffusion equation (Fick's law). Molecular

diffusion coefficients are properties of the solvent and solute and are almost constant. Typically they are of the order of $10^{-9}\,m^2\,s^{-1}$.

2. In turbulent flow a tracer mixes more rapidly than can be accounted for just by molecular diffusion. The reason is that small-scale turbulent eddies continually erode the edges of the tracer cloud, increase local concentration gradients and hence allow molecular diffusion to proceed more rapidly. The combined effect of molecular diffusion and turbulent velocity fluctuations is termed turbulent diffusion. In stationary homogenous turbulence, turbulent diffusion can be modelled satisfactorily using Fick's law provided sufficient time has elapsed since the time of injection. Thus asymptotically the tracer flux resulting from turbulent diffusion is proportional to the spatial gradient of the ensemble mean tracer concentration. In the $x$ direction

$$J_x = -e_m \frac{\partial \langle c \rangle}{\partial x}$$

where $J_x$ = mass flux in the $x$ direction; $\langle c \rangle$ = ensemble mean concentration and $e_m$ = turbulent diffusion coefficient. The ensemble mean concentration is the average concentration which would be expected over a large number of identical tracer experiments. Turbulent diffusion coefficients are properties of the flow and are not constant. They vary from channel to channel and with location and direction within a given channel. Typically turbulent diffusion coefficients are of the order of $10^{-3}\,m^2\,s^{-1}$.

3. In the most general problem, advection and turbulent diffusion occur in each of the three coordinate directions. The three-dimensional advection/diffusion equation (in advection form) is

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} + u_z \frac{\partial c}{\partial z} = \frac{\partial}{\partial x}\left(e_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(e_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(e_z \frac{\partial c}{\partial z}\right)$$

where $x$, $y$ and $z$ = longitudinal, vertical and transverse distances; $c$ = ensemble mean tracer concentration; $u_x$, $u_y$ and $u_z$ = velocities; and $e_x$, $e_y$ and $e_z$ = turbulent diffusion coefficients.

4. The advection/diffusion equation describes the behaviour of the ensemble mean concentration. Because of the random nature of turbulence, concentrations fluctuate about the ensemble mean. At present it is not possible to predict extreme concentrations theoretically but empirical methods can be used to establish parameters such as the peak to mean concentration ratio.
5. In the near-field mixing region close to a non-steady point source, advection and mixing are important in all three coordinate directions and the full three-dimensional advection/diffusion equation is required to predict tracer concentrations. In some near-field mixing problems, however, the advection/diffusion equation can be simplified. For example, if the source is steady then longitudinal concentration gradients become negligibly small. If the $x$ axis is aligned with the flow then velocities $u_y$ and $u_z$ can be neglected. If, in addition, the tracer originates from a steady vertical line source then vertical concentration gradients are small everywhere and attention can be focused on transverse mixing. On the other hand if the tracer originates from a steady transverse line source then transverse concentration gradients are small everywhere and attention can be focused on vertical mixing. Simplification of the mixing equations must be done with great care to retain the important terms.



7. In most rivers, tracer originating from a point source mixes vertically long before it mixes transversely. In the mid-field mixing region vertical concentration gradients are small and may not be important ecologically. The full three-dimensional advection/dispersion equation can be averaged over the depth to yield the two-dimensional equation

$$\frac{\partial s}{\partial t} + v_x \frac{\partial s}{\partial x} + v_z \frac{\partial s}{\partial z} = \frac{1}{h}\frac{\partial}{\partial x}\left(hk_x \frac{\partial s}{\partial x}\right) + \frac{1}{h}\frac{\partial}{\partial z}\left(hk_z \frac{\partial s}{\partial z}\right)$$

where $s$ = depth-averaged tracer concentration; $v_x$ and $v_z$ = depth-averaged velocities; $h$ = local depth; and $k_x$ and $k_z$ = longitudinal and transverse dispersion coefficients. The dispersion coefficients quantify the effects on tracer mixing of vertical variations in velocity.

8. Tracer originating from a non-steady point source eventually mixes across the channel. In the far-field vertical and transverse concentration gradients may not be important ecologically and so the three-dimensional advection/diffusion equation can be averaged over both the depth and the width to yield the one-dimensional equation

$$\frac{\partial S}{\partial t} + V_x \frac{\partial S}{\partial x} = \frac{1}{A}\frac{\partial}{\partial x}\left(AK_x \frac{\partial S}{\partial x}\right)$$

where $S$ = cross-sectional averaged concentration; $V_x$ = cross-sectional averaged velocity; $A$ = cross-sectional area; and $K_x$ = longitudinal dispersion coefficient.

9. Figure 1.9 shows three types of tracer source and sketches the concentration distributions observed at different distances below each one. Table 1.2 gives approximate formulae for estimating the length of the near-field and mid-field mixing regions and also gives the dimensionality of the mixing equations in each region.
10. Averaging over the depth and/or width reduces the dimensionality of the advection/diffusion equation and simplifies its solution. When averaging is performed it is important to account for the effects of non-uniformities of velocity in the coordinate directions over which averaging is being carried out. Non-uniformities of velocity (velocity shear) contribute to the spreading of tracer and when the advection/diffusion equation is

averaged, their effects must be quantified by additional dispersion terms. Dispersion is not a fundamental physical process but arises as an artifact of averaging. The correct modelling of dispersion plays an important part in the development and use of mixing models. Under certain conditions Fick's law can be used to model dispersion. The most important requirement is that sufficient time must have elapsed since tracer injection for an equilibrium to have become established between velocity shear and turbulent diffusion.

11. If the channel is straight and uniform then it is best to write the mixing equations in rectangular Cartesian coordinates. In most natural rivers, however, the channel meanders and varies in shape and it is natural to rewrite the mixing equations in curvilinear coordinates. The curvilinear equations can be simplified by replacing transverse distance by cumulative flow, thereby giving the streamtube model.

## 1.3 CONTINUUM HYPOTHESIS

Before discussing mixing processes we need to have a clear conceptual picture of liquids and dissolved tracers. At very small length scales (typically a few angstroms) a solution consists of a mixture of discrete solvent and solute molecules. At these very small length scales variables such as velocity and tracer concentration have violently non-uniform distributions which are difficult to handle mathematically. Attempts to describe the mechanics of liquids at the molecular level using statistical methods have not been very successful (Batchelor 1967). In river mixing problems, however, the smallest length scales of interest are much larger than the distance between molecules. It is convenient to adopt the hypothesis that the fluid is a continuum in which velocities and tracer concentrations are some sort of average of the values at the molecular level and are well-behaved functions of time and location. This is known as the continuum hypothesis. One consequence of the continuum hypothesis is that when analysing turbulent flows we cannot model explicitly variations of velocity and concentration at molecular length scales. This does not present major problems because it has been found that it is the velocity variations with large length scales (i.e. the large eddies) which effect most of the momentum and mass transport in turbulent flows (Townsend 1956, Tennekes and Lumley 1972).

Within the continuum framework it is often useful to imagine a small parcel of fluid whose dimensions are comparable with the smallest length scale of interest and which has characteristic properties such as velocity and tracer concentration. A collection of such parcels is termed a tracer cloud.



## 1.4 ADVECTION

Advection is the bodily movement of a parcel of fluid resulting from an imposed current. Advection transports any tracer which may be dissolved or suspended in the fluid and is clearly important in rivers because it carries tracer downstream away from a fixed source. Some workers use the term convection to describe this process (e.g. Holley 1969) but this terminology is potentially confusing and is not used here because in the atmospheric and geothermal publications convection implies a flow induced by buoyancy. In rivers water moves largely as a result of gravity (channel slope) and buoyancy has little effect, except sometimes very close to a source of cooling water or dense tracer.

The amount of tracer transported per unit time per unit area perpendicular to the current is termed the advective flux and is the product of the velocity and the tracer concentration. Considering a small parcel of fluid and transport in the $x$ direction

$$I_x = u_x c \tag{1.1}$$

where $I_x(x, y, z, t)$ = advective flux in the $x$ direction; $u_x(x, y, z, t)$ = velocity in the $x$ direction; and $c(x, y, z, t)$ = tracer concentration. Under the continuum hypothesis $u_x$, $c$ and hence $I_x$ are assumed to have well-defined values for each parcel of fluid. Note that $I_x$ varies with both location and time.

It is important to appreciate that pure advection does not cause any change of concentration within a tracer cloud. To illustrate this point, consider fluid parcels in a flow where the velocity is uniform (does not vary with $x$, $y$ or $z$). If we ignore molecular diffusion then advection causes the fluid parcels to move bodily downstream at a constant rate without undergoing any distortion or spreading. This is, of course, a hypothetical example because uniform flow rarely occurs in nature and the tracer cloud is affected by molecular and turbulent diffusion. A Lagrangian coordinate system is one which travels downstream at the mean velocity (as opposed to a Eulerian system which remains fixed in space). We often use a Lagrangian coordinate system whose origin lies at the centre of a tracer cloud. The advantage of a Lagrangian coordinate system is that concentration changes only as a result of diffusion and dispersion and not as a result of advection. This holds for both steady and unsteady tracer sources. By comparison, in an Eulerian coordinate system advection causes the concentration at a fixed point downstream from an unsteady source to vary with time, which complicates the analysis.

## 1.5 MOLECULAR DIFFUSION

### 1.5.1 Molecular Diffusion in a Stagnant Fluid

If neutrally buoyant tracer is introduced very carefully into stagnant water at a point far distant from any boundaries then it will be seen to spread out slowly at the same rate in all three coordinate directions. This spreading results from random molecular motion within the fluid (the so-called Brownian motion) and is termed molecular diffusion. Attempts to calculate the rates of molecular diffusion theoretically have been unsuccessful for solids and fluids and have met with only limited success in gases. In 1855, however, the German physiologist Fick published a paper in which he drew an analogy between the diffusion of salt in water with the diffusion of heat along a metal rod. He made the hypothesis that the net rate of transfer of tracer from a region of high concentration to a region of low concentration should proceed at a rate proportional to the concentration gradient between the two regions. This has become known as Fick's first law. It is a particular case of the more general gradient diffusion law which governs the flow of electricity along a conducting wire (Ohm's law) and the flow of heat along a metal rod (Fourier's law). In one dimension Fick's first law can be written

$$J_x = -e_m \frac{\partial c}{\partial x} \tag{1.2}$$

where $J_x$ = molecular diffusive flux in the $x$ direction (i.e. the transfer rate of tracer in the $x$ direction per unit area by molecular diffusion); $c$ = tracer concentration; $\partial c/\partial x$ = tracer concentration gradient in the $x$ direction; and $e_m$ = molecular diffusion coefficient. The negative sign in equation (1.2) arises because tracer diffuses from a region of high concentration to a region of low concentration.

The gradient diffusion law is based on a hypothesis rather than on a complete theoretical analysis but laboratory studies show that the relationship between flux and the concentration gradient is remarkably linear. The constant of proportionality between flux and concentration gradient, the molecular diffusion coefficient (also termed the molecular diffusivity), must of course be determined empirically. Typical values of $e_m$ for solutes in water are in the range $0.5–2.0 \times 10^{-9}\,m^2\,s^{-1}$. The molecular diffusion coefficient is a property of the fluid and for a given solvent, solute, concentration and temperature, $e_m$ is constant. In contrast turbulent diffusion coefficients (which are discussed shortly) are much larger than molecular diffusion coefficients and are properties of the flow rather than the fluid.

### 1.5.2 Molecular Diffusion in Laminar Flow

We now consider molecular diffusion in steady laminar flow. In laminar flow viscous forces are strong relative to inertial forces and fluid elements follow smooth paths (streamlines) which are reproducible and do not cross each other. The balance between viscous and inertial forces can be quantified using the Reynolds number which is defined

$$Re = \frac{uL}{\nu} \tag{1.3}$$

where $u$ = a characteristic velocity; $L$ = a characteristic length scale; and $\nu = \mu/\rho$ = the kinematic viscosity of the fluid. At low Reynolds numbers the flow is laminar, at high numbers it is fully turbulent and at intermediate numbers it is transitional. The appropriate length and velocity scales and the critical Reynolds numbers depend on the nature of the mixing problem. In pipe flow $u$ = mean velocity, $L$ = pipe diameter, flows are laminar provided the Reynolds number is less than about 2000 and fully turbulent above about 10 000. In river channels $u$ = mean velocity, $L$ = mean depth, flows are laminar at Reynolds numbers below 500 and turbulent above about 2000. For boundary layers $u$ = free stream velocity, $L$ = boundary layer thickness and the boundary layer remains laminar at Reynolds numbers below about 600.

To model tracer concentrations, we need to combine the effects of advection and molecular diffusion. This is done by assuming that the two processes are independent and additive. A Lagrangian coordinate frame of reference is adopted which moves with the fluid at the mean velocity. Within this Lagrangian coordinate system molecular diffusion is assumed to occur as if the fluid were stationary and hence the diffusive flux is given by the product of the concentration gradient (measured in the Lagrangian framework travelling at the mean velocity) and the molecular diffusivity. Having made this assumption it is possible to derive the equation governing tracer concentration based on the conservation of mass.

### 1.5.3 Derivation of the Molecular Diffusion Equation

We shall adopt a rectangular Cartesian coordinate system with axes $x$, $y$ and $z$. Imagine a very small rectangular parcel of fluid which moves at the mean velocity (Figure 1.1). Conservation of mass requires that the rate of change of mass within this parcel is balanced by the rates of transfer of mass across the boundaries. Note that advection does not transport any material across the parcel boundaries because the parcel is travelling at the mean velocity. Thus

$$\frac{M_{t+\Delta t} - M_t}{\Delta t} = (J_x - J_{x+\Delta x})\Delta y \Delta z + (J_y - J_{y+\Delta y})\Delta x \Delta z + (J_z - J_{z+\Delta z})\Delta x \Delta y \tag{1.4}$$

where $M_t$ and $M_{t+\Delta t}$ tracer mass in the moving parcel at times $t$ and $t+\Delta t$ respectively; $J_x$, $J_y$ and $J_z$ diffusive fluxes (averaged over the time interval

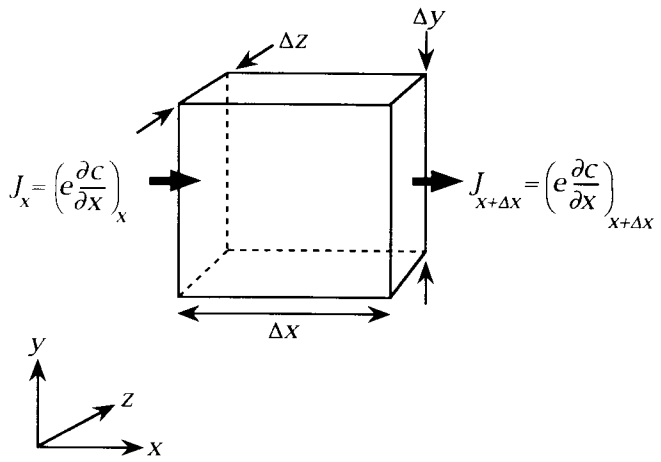


**Figure 1.1.** Diffusive fluxes into and out of a small fluid element

$\Delta t$) entering the parcel through the moving boundaries located at $x$, $y$, $z$; $J_{x+\Delta x}$, $J_{y+\Delta y}$ and $J_{z+\Delta z}$ = diffusive fluxes (averaged over the time interval $\Delta t$) leaving the parcel through the moving boundaries located at $x+\Delta x$, $y+\Delta y$, $z+\Delta z$; $\Delta x$, $\Delta y$ and $\Delta z$ = dimensions of the fluid parcel; and $\Delta t$ = time interval. Using a Taylor's series expansion and ignoring second order and higher terms (because $\Delta t$ is very small)

$$M_{t+\Delta t} = M_t + \left(\frac{DM}{Dt}\right)_t \Delta t \tag{1.5}$$

Note that in equation (1.5) we use the material operator $D/Dt$ to denote the rate of change with time in a Lagrangian coordinate system travelling at the mean velocity. The mass of tracer within the parcel at time $t$ is

$$M_t = c_t \Delta x \Delta y \Delta z \tag{1.6}$$

where $c_t$ = average tracer concentration within the parcel at time $t$. Consequently equation (1.5) can be rewritten

$$\frac{M_{t+\Delta t}}{\Delta t} - M_t = \left(\frac{Dc}{Dt}\right)_t \Delta x \Delta y \Delta z \tag{1.7}$$

Now we need to evaluate the diffusive flux into and out of the moving parcel. Because diffusion is defined in a Lagrangian coordinate system then strictly we need to know the concentration gradients across each of the parcel boundaries as they move downstream. These gradients are difficult to evaluate. As the time increment $\Delta t$ is very small, however, we can assume that the parcel is at rest. The concentration gradients across the boundaries can



then be determined using Eulerian operators. Thus

$$J_{x+\Delta x} = J_x + \left(\frac{\partial J_x}{\partial x}\right)_x \Delta x \tag{1.8a}$$

$$J_{y+\Delta y} = J_y + \left(\frac{\partial J_y}{\partial y}\right)_y \Delta y \tag{1.8b}$$

$$J_{z+\Delta z} = J_z + \left(\frac{\partial J_z}{\partial z}\right)_z \Delta z \tag{1.8c}$$

In the limit as $\Delta t$, $\Delta x$, $\Delta y$ and $\Delta z$ go to zero these approximations are valid and present no difficulties when using analytical solutions to the resulting advection/diffusion equation. It is worth noting, however, that problems with numerical dispersion and oscillatory solutions can arise when using discrete numerical methods to solve the advection/diffusion equation. These problems stem from the fact that diffusion is defined in a moving Lagrangian coordinate system whereas advection is defined in a fixed Eulerian coordinate system.

Using equation (1.2) to evaluate the diffusive fluxes in equation (1.8) and rearranging

$$(J_x - J_{x+\Delta x})\Delta y \Delta z = \frac{\partial}{\partial x}\left(e_m \frac{\partial c}{\partial x}\right)\Delta x \Delta y \Delta z \tag{1.9a}$$

$$(J_y - J_{y+\Delta y})\Delta x \Delta z = \frac{\partial}{\partial y}\left(e_m \frac{\partial c}{\partial y}\right)\Delta x \Delta y \Delta z \tag{1.9b}$$

$$(J_z - J_{z+\Delta z})\Delta x \Delta y = \frac{\partial}{\partial z}\left(e_m \frac{\partial c}{\partial z}\right)\Delta x \Delta y \Delta z \tag{1.9c}$$

Combining equations (1.4), (1.7) and (1.9), noting that $e_m$ is independent of $x$, $y$ and $z$ and simplifying gives

$$\frac{Dc}{Dt} = e_m\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{1.10}$$

The material derivative is (Batchelor 1967)

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} + u_z \frac{\partial c}{\partial z} \tag{1.11}$$

Equation (1.10) can now be rewritten to yield the well-known three-dimensional Fickian diffusion equation in rectangular Cartesian coordinates

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} + u_z \frac{\partial c}{\partial z} = e_m\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{1.12}$$

Provided the velocities and the molecular diffusion coefficient are known and suitable initial and boundary conditions for concentrations can be specified,

then equation (1.12) can be solved to predict the changes of tracer concentration with time and location.

### 1.5.4 Properties of the Molecular Diffusion Equation

A very important solution to equation (1.12) describes the spreading of a conservative tracer slug of mass $M$ released at $x = y = z = 0$ and $t = 0$ in an unbounded channel. If $u_x$, $u_y$, $u_z$ and $e_m$ are constant then

$$c(x, y, z, t) = \frac{M}{(4\pi e_m t)^{3/2}} \exp\left( - \frac{(x - u_x t)^2 + (y - u_y t)^2 + (z - u_z t)^2}{4 e_m t} \right) \quad (1.13)$$

At a given time the tracer concentration predicted using equation (1.13) is distributed along the $x$, $y$ and $z$ axes in a Gaussian bell-shaped curve (Figure 1.2). As will be discussed later, in field and laboratory studies tracer distributions are often measured which are approximately Gaussian and equations such as (1.12) and (1.13) can be used to model such data.

Another important feature of equations (1.12) and (1.13) relates to the spatial variance of the tracer distribution. The spatial variance in the $x$ direction is defined

$$\sigma_x^2(t) = \frac{1}{M} \iiint (x - \bar{x})^2 c(x, y, z, t) \, dx \, dy \, dz \quad (1.14a)$$

$$\bar{x}(t) = \frac{1}{M} \iiint x c(x, y, z, t) \, dx \, dy \, dz \quad (1.14b)$$

$$M = \iiint c(x, y, z, t) \, dx \, dy \, dz \quad (1.14c)$$

where $\bar{x} = x$ coordinate of the centroid of the tracer distribution and $M$ = tracer mass. The variances in the $y$ and $z$ directions are defined in an analogous fashion. It is a straightforward exercise to show that for equation (1.13)

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 2 e_m t \quad (1.15)$$

It is a feature of solutions to equation (1.12) that the variance increases linearly with time. In field studies it is often observed that the variance of observed tracer distribution increases with time at a linear rate, which suggests that such data can be modelled by equations such as (1.12).

Tracer distributions do not need to be Gaussian for the variance to increase linearly with time. Consider an experiment where equation (1.12) applies but the initial tracer distribution along the $x$ axis is markedly skewed because of the way the tracer was introduced into the channel. During the initial phases of the experiment the tracer distribution will be skewed but the variance of the



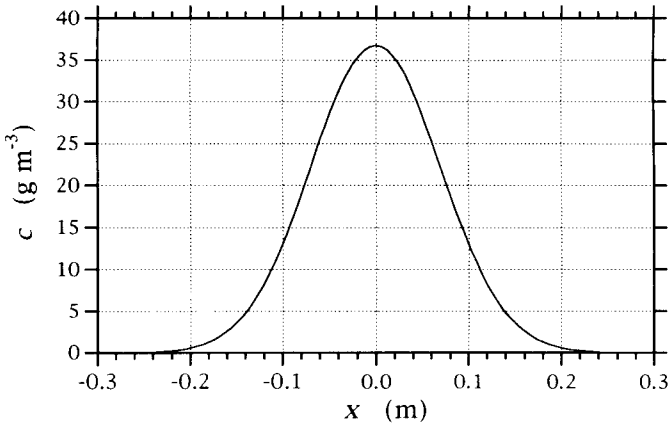


**Figure 1.2.** Tracer distribution along the $x$ axis at $y = z = 0$ predicted using equation (1.13) with $M = 1$ g, $t = 1000$ hours, $e_m = 2 \times 10^{-9}$ m$^2$ s$^{-1}$ and $u_x = u_y = u_w = 0$

tracer distribution will always increase linearly with time. The skewness of the tracer distribution will decrease gradually with time and at asymptotically long times the distribution will become Gaussian.

The linear increase of variance is a necessary but not a sufficient condition for the validity of equation (1.12). In other words there may be other more complex diffusion equations whose solutions show linear growth of variance. One such model is discussed briefly in Chapter 4 (see equation 4.22). There have been several theoretical investigations of non-Fickian diffusion models but few, if any, of these other equations are used in practical mixing problems.

A third important feature of equation (1.12) is that it is linear so that solutions for a number of slug injections can be superimposed to construct solutions to more complex problems. Superposition is discussed in more detail in Chapter 3 (Section 3.7).

Finally, it is worth noting that there are analytical solutions to equation (1.12) available for a large number of heat conduction problems (Carslaw and Jaegar 1959, Crank 1975) which can be adapted readily for analysing diffusion problems. A summary of these is given in the Appendix.

## 1.6 TURBULENCE

### 1.6.1 Features of Turbulent Flow

Most river and open channel flows are characterised by high Reynolds numbers [see equation (1.3)] and are turbulent except in a very thin, viscous sub-layer near the bed (there is a more detailed discussion of the viscous

sub-layer in Chapter 6). One important feature of turbulent flow is that tracer spreads far more rapidly than in laminar flow. In turbulent flow viscous forces are weak relative to inertial forces. The velocity at a point exhibits random short-term fluctuations even if the long-term average velocity is constant. This is in marked contrast with laminar flow in which the velocity at a point varies smoothly and predictably over time. Particles released into a turbulent flow follow highly irregular paths and particles released from the same point at different times may follow significantly different paths. By comparison particles released from the same point in a steady laminar flow will follow the same path.

Turbulent flows are not, however, entirely chaotic and velocities at adjacent locations are correlated. The correlations are very strong if the points are close together but decrease as the separation between points increases until at points a long way apart the correlations become negligibly small. Spectral analysis is used to quantify the correlation between velocities at two (or more) points as a function of the distance separating the points. A typical spectrum may have pronounced peaks at given separations, say $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$, and this gives rise to the concept that within the flow there are eddies whose characteristic length scales are $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$. Everyone is familiar with the large-scale eddies which disturb the water surface in flowing water. Eddies also become obvious when a tracer such as dye is introduced into the flow from a steady point source. The largest separation at which the velocities remain correlated provides an estimate of the largest turbulent length scale within the flow and hence the size of the largest eddies. In rivers the length scales of the largest vertical and horizontal eddies cannot be larger than, and are often found to be comparable with, the channel depth and the channel width, respectively.

Turbulent flows are not self-sustaining and in the absence of an external energy input, turbulence decays and the flow reverts to being laminar. In rivers the energy input is provided by gravity (channel slope), although very close to an outfall it may be provided by a pressure gradient. In rivers turbulence is generated by velocity shear (Tennekes and Lumley 1972) and hence originates in regions where there are strong velocity gradients, notably near the bed, banks and obstacles within the flow. Energy enters the flow at fairly large length scales but there are strong non-linearities in the Navier–Stokes equations governing turbulent flow which cause energy to be transferred to smaller eddies. This gives rise to a wide range of eddy sizes within turbulent flow. Eventually the energy reaches the smallest eddies and is dissipated as heat. The smallest length scales present in turbulent river flows have been shown to be of the order $10^{-4}$ to $10^{-3}$ m (Fischer *et al* 1979, Chatwin and Allen 1985). As is intuitively obvious it tends to be the largest eddies which effect most of the momentum and mass transfer in turbulent flows (Townsend 1956, Tennekes and Lumley 1972).

Turbulence is termed stationary if characteristics such as the mean velocity,



velocity variance and the correlations between velocities remain constant with time (in turbulent flow stationarity is the equivalent of steadiness in laminar flow). Turbulence is homogeneous if the velocity fluctuations and correlations do not vary with location and is isotropic if they are the same in all coordinate directions. A body of theory has been developed for diffusion in stationary, homogeneous turbulence (Batchelor 1953, Townsend 1956). Turbulence in natural channels is seldom homogeneous or stationary and consequently theoretical results cannot always be applied directly. Despite this difficulty the theoretical results provide insights into the form which the turbulent mixing equations should take and they also help to guide experimental work.

### 1.6.2 Role of Averaging in Turbulent Flows

It is necessary to make a slight digression at this point to discuss the role of averaging in studies of turbulent flows. Consider again equation (1.12). This equation is exact and in theory it could be used to model tracer concentrations in turbulent flow. We would need to know how the velocity varies with time and space (down to very small length and time scales) and to have suitable initial and boundary conditions. Given this information and a very large computer we could try and predict the movement of each separate fluid parcel. From time to time we could determine the average tracer concentration over any length scale we were interested in and hence could determine the average concentration of the tracer cloud. An important feature of turbulent flow, however, is that velocities vary randomly with time and it is simply not practicable to try and make deterministic predictions of velocity fluctuations or to make sufficient velocity measurements to address complex mixing problems in this way. We can, however, make the problem more tractable by examining the behaviour of certain space and time averages of velocity, tracer concentration and advective flux.

There are three different sorts of average which are used when studying turbulence: ensemble, time and space averages. Consider first the ensemble average. Imagine conducting an experiment in which a known mass of tracer is released and the tracer concentration is measured at a fixed point $x_o$, $y_o$, $z_o$ and time $t_o$ after injection. Now repeat the same experiment several times, each time measuring the concentration at the same point and at the same time after release. Because the flow is turbulent each concentration measurement will be different, giving rise to a series of concentrations $c_i(x_o, y_o, z_o, t_o)$ where $i = 1, 2, \ldots N$ and $N$ is the total number of experiments. Then averaging over all the experiments yields an estimate of the ensemble average concentration at $x_o$, $y_o$, $z_o$ and $t_o$.

$$\langle c(x_o, y_o, z_o, t_o) \rangle = \frac{1}{N} \sum_{i=1}^{i=N} c_i(x_o, y_o, z_o, t_o) \qquad (1.16)$$

The angled brackets $\langle\ \rangle$ are customarily used to denote an ensemble average. Thus an ensemble average is taken over a large number of independent experiments or realisations and can be calculated for concentration, velocity or various velocity and concentration products.

In some situations the statistical properties of the concentration are independent of time over some time window $T$. Tracer concentrations below a steady source are an example. In such situations the ensemble average concentration at a fixed point can be approximated by the time average

$$\langle c(x_o, y_o, z_o, t_o)\rangle \approx \frac{1}{T}\int_{\tau=0}^{T} c(x_o, y_o, z_o, \tau)\,d\tau \qquad (1.17)$$

where $0 < \tau < T$. Alternatively, in a region where the tracer is well-mixed but rapidly time-varying, it may be valid to approximate the ensemble average by the spatial average of the concentration over the volume $V$.

$$\langle c(x_o, y_o, z_o, t_o)\rangle \approx \frac{1}{V}\iiint c(x, y, z, t_o)\,dx\,dy\,dz \qquad (1.18)$$

where $x_o$, $y_o$, $z_o$ lies within the volume $V$. The assumption that the ensemble, time and space averages are identical is known as the ergodic hypothesis.

Theoreticians insist that the ensemble average should always be used when analysing turbulent mixing (Lumley and Panofsky 1964). The main reason is that separate realisations are independent so that ensemble averages are unbiased. By comparison measurements made at small intervals of space and time may not be independent and hence the resulting average may not be a reliable estimate of the true mean value. When addressing practical problems, however, it is not always feasible to conduct the large number of trials which would seem to be required to carry out ensemble averaging and consequently time and space averaging are often used despite their theoretical shortcomings. It is clearly desirable to make the averaging period (or volume) as large as possible to reduce the possibility of bias.

In situations where the period of averaging $T$ is large compared with the time-scale of the turbulent fluctuations in concentration then equation (1.17) can be expected to provide a reliable estimate of the ensemble average. If, on the other hand, $T$ is comparable with or smaller than the time-scale of turbulent fluctuations, then the ergodic hypothesis is invalid and equation (1.17) will yield poor estimates of the ensemble average concentration. This can present a serious problem in estuarine flows where it is desirable to make the averaging period $T$ as large as possible, but tidal action restricts the time window over which the underlying characteristics of the flow are constant. In estuaries the maximum averaging period which can safely be used is $T = 60$–$90$ seconds (Chatwin and Allen 1985). Fortunately this problem arises less frequently when dealing with a steady river flow and steady tracer input. The same



difficulty arises with spatial averaging. To obtain an unbiased estimate of the space-averaged concentration it is necessary to collect a number of samples at sufficiently large separation to ensure that they are independent. This implies sampling over a large area, but in many practical problems there may not be a large area where the concentration is constant except for random turbulent fluctuations. It will always be possible to determine a space or time average from experimental data but it is important to remember that these estimates may pertain only to the particular set of conditions sampled and may not give a robust estimate of typical conditions.

### 1.6.3 Extreme Concentrations

When assessing the ecological impact of a toxic pollutant the average concentration is clearly important but some knowledge of concentration extremes is also desirable. As pointed out by Fischer *et al* (1979), if a few high concentrations kill an organism then even a large number of low concentrations will not bring it back to life. Ideally we would like to be able to predict the probability density function of the concentration at various points in the flow because we could then say something about both mean and extreme concentrations. If, in addition, we know the time spectrum of the concentration fluctuations, then we also know something about how long high and low concentrations persist. Unfortunately the existing models for turbulent flows predict only the average concentration and do not furnish estimates of the probability density functions or the time spectra of concentrations, although current research is addressing this topic. The practitioner must rely on empirical studies to give an indication of the concentration fluctuations likely to be encountered. Much of the

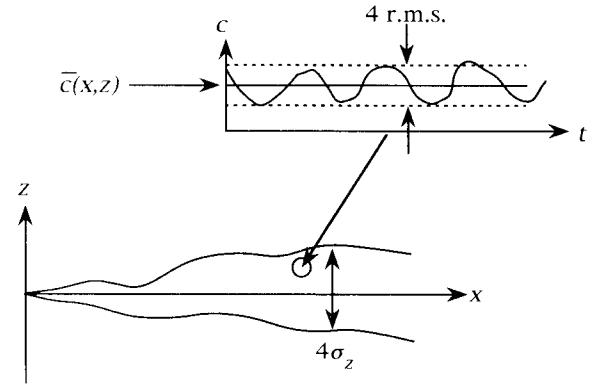


**Figure 1.3.** Sketch of the plume downstream from a steady point source in turbulent flow showing how concentration measured at a fixed point fluctuates with time

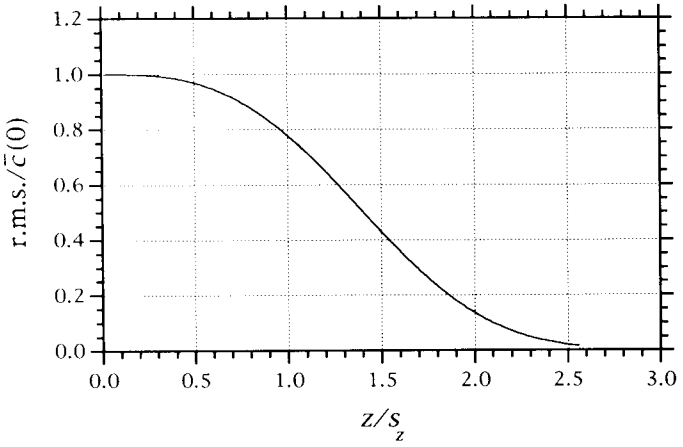


**Figure 1.4.** R.m.s. deviation of tracer concentration fluctuations over time as a function of distance from the centreline typical of experimental measurements downstream from a steady point source. The r.m.s. deviation is non-dimensionalised using the mean centreline concentration and the transverse distance is non-dimensionalised using the transverse spatial variance of the plume

experimental work on concentration extremes has been done in the atmosphere, but similar mechanisms operate in air and water.

Consider the problem of concentrations in the $x$–$z$ plane downstream from a steady point source discharging tracer into an unbounded channel with flow in the $x$ direction (see Figure 1.3). A continuously recording probe placed at a fixed location within the plume measures concentrations $c_i$ where $i = 1, 2, 3 \ldots N$ at short intervals of time. The mean and root mean square (r.m.s.) deviation[1] over time are calculated at a number of points across the plume at a fixed distance downstream from the source. Figure 1.4 shows the typical distribution which is found in experimental studies. The r.m.s. deviation is normalised using the mean centreline concentration and the transverse distance is normalised using plume width (here taken as the transverse standard deviation[2] of the plume). At the centreline of the plume the r.m.s.

[1] The r.m.s. deviation is defined

$$\text{r.m.s.} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(c_i - \bar{c})^2} \qquad \text{where} \qquad \bar{c} = \frac{1}{N}\sum_{i=1}^{N} c_i$$

[2] The transverse standard deviation is defined

$$\sigma_z^2 = \frac{\int_{-\infty}^{\infty}(z - \bar{z})^2 c \, dz}{\int_{-\infty}^{\infty} c \, dz} \qquad \text{where} \qquad \bar{z} = \frac{\int_{-\infty}^{\infty} z \, dz}{\int_{-\infty}^{\infty} c \, dz}$$



deviation over time attains its maximum value which is comparable with the mean centreline concentration. Either side of the centreline the absolute value of the r.m.s. deviation decreases and it vanishes outside the plume. Csanady (1967) has gone some way towards predicting the distribution of the r.m.s. deviation within the plume.

Knowing the r.m.s. deviation we are able to say something about the maximum concentration. The absolute maximum concentration is not easy to measure, predict or interpret and a more useful measure is the concentration which is only exceeded a certain percentage of the time (say 10, 5 or 1%). The probability density function of the concentration is found to be skewed because concentrations must be non-negative and there are a few very high concentrations. This type of data can be described by either a log-normal or a Poisson distribution (Csanady 1973). For the Poisson distribution the concentrations exceeded 10 and 1% of the time are, respectively, 2.3 and 4.6 times the mean concentration (Freund 1974). Thus near the centreline of the plume we would expect the peak to mean ratio to lie in the range two to five. This accords well with experimental results in atmospheric and lake studies in which the peak to mean ratio is typically three (Csanady 1967). It might be unwise to assume that tracer plumes in rivers behave in exactly the same way because the flows are rather different, but clearly the equivalent of Figure 1.4 could be derived from experimental results in rivers.

Figure 1.4 indicates that towards the edge of the plume the ratio of the r.m.s. deviation to the centreline concentration decreases. The local concentration also decreases and it is important to understand how the peak to mean ratio behaves locally. Experimental work in the atmospheric boundary layer shows that the local peak to mean ratio increases from about three at the centreline (as discussed in the previous paragraph) to about 20–50 at the edge of the plume, with occasional values as high as 100. These ratios are high at the edge of the plume because for much of the time there is no tracer present, but occasionally an eddy containing an appreciable amount of tracer is present at the sampling point for a short period of time. It is found in atmospheric studies that the peak to mean ratio at the edge of the plume is highest close to the source and decreases with distance downstream so that at very large distances from the source the peak to mean ratio reaches a lower limit of the order of two. Again it must be stressed that these figures may not be directly applicable to river mixing problems. It is not immediately obvious why the peak to mean ratio should decrease with distance from the source. Presumably close to a point source the plume contains a small number of fluid parcels containing high tracer concentrations among other parcels substantially free from tracer. Further away from the source diffusion has had time to mix the tracer between adjacent fluid parcels so that concentrations are less variable across the plume.

In the remainder of this book we will discuss models and coefficients for predicting the ensemble mean tracer concentration. It is important to

remember, however, that the tracer concentration will fluctuate about this mean value. An initial assessment can be made of the importance of these fluctuations using the ideas and results discussed in the preceding paragraphs but to make more accurate predictions of concentration fluctuations it will be necessary to conduct site-specific experimental studies.

*Example 1.1*

Short-term temperature fluctuations measured in the Waikato River, New Zealand, downstream from the Huntly power station. Cooling water with an excess temperature averaging $+8^{\circ}C$ is discharged from a bankside outfall which mixes the effluent with 50–80% of the river flow within about 300 m of the outfall. Vertical mixing is complete within 200–300 m. Temperatures were measured every second for 30 minutes at four sections 250–550 m downstream from the outfall. Results are summarised in Table 1.1.

Temperature fluctuations were found to have a periodicity of the order of three minutes at each site. The distribution of temperature was approximately normal and the peak to mean ratio was found to lie in the range 1.05–1.12. This is markedly different from the experimental results for a point source discussed in the text where the tracer distribution is skewed and the minimum peak to mean ratio is typically three. The difference arises because the Huntly cooling water has a maximum excess temperature of $+8^{\circ}C$ at the point of discharge (which places an upper limit on the peak to mean ratio) and there is rapid vertical mixing together with significant entrainment of ambient river water close to the outfall.

For measurements 40 m from the left bank (where temperatures are close to the maximum) the peak to mean ratio decreased with distance downstream from the outfall Table 1.1. This is in accord with the experimental results discussed in the text. For measurements made near the edge of the plume (70 m from the left bank) the absolute value of the r.m.s. deviation was nearly twice that measured at the other sites and the local peak to mean ratio was also significantly higher. Along the edge of the plume there is a high temperature gradient and turbulent eddies in this region cause larger temperature flucuations than elsewhere in the plume where the temperature gradients are smaller.

**Table 1.1.** Temperature fluctuations with time measured in the Waikato River below the Huntly power station. During Experiments 1 and 2 initial dilution was with 80 and 45% of river flow, respectively. Transverse distances are measured from the left bank. The peak temperature occurs close to the left bank. The r.m.s. deviation is calculated over 30 minutes. All temperatures are $^{\circ}C$ above ambient

| Longitudinal distance (m) | Transverse distance (m) | Transect peak temperature ($^{\circ}C$) | Local mean temperature ($^{\circ}C$) | r.m.s. time deviation ($^{\circ}C$) | Peak to mean ratio |
|---|---|---|---|---|---|
| Expt. 1 | | | | | |
| 250 | 40 | 2.3 | 2.0 | 0.063 | 1.071 |
| 350 | 40 | 2.2 | 2.0 | 0.050 | 1.058 |
| 550 | 40 | 2.2 | 2.0 | 0.053 | 1.053 |
| Expt. 2 | | | | | |
| 450 | 40 | 4.5 | 3.0 | 0.063 | 1.048 |
| 450 | 70 | 4.5 | 2.0 | 0.100 | 1.115 |



### 1.6.4 Lagrangian Coordinate Systems

It is clear from the derivation of equation (1.12) that it is best to describe diffusion in a Lagrangian coordinate system which moves at the mean velocity. In steady laminar flow there is little difficulty defining the mean velocity to use. In stationary turbulent flow, however, there are two possible Lagrangian coordinate systems to choose from: one which moves at the mean velocity of an individual tracer cloud and one which moves at the ensemble mean velocity.

Imagine conducting a number of identical experiments in each of which an instantaneous tracer release is made. Figure 1.5 shows two tracer clouds each measured at time $t$ after release. Because of the random nature of turbulence, each tracer cloud follows a slightly different path and spreads in a slightly different manner. The broken axes in Figure 1.5 are placed through the

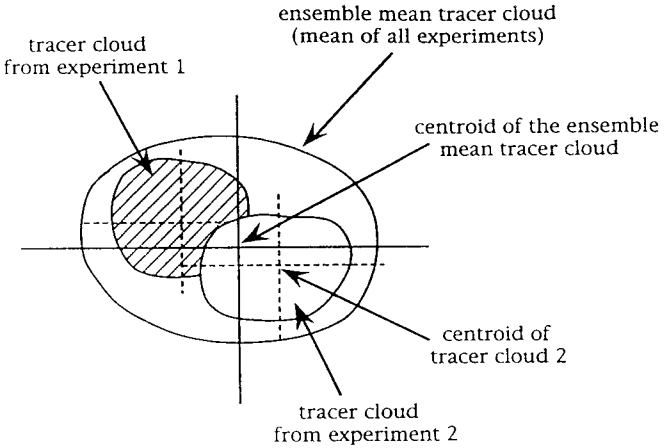


**Figure 1.5.** Sketch showing tracer clouds from two different experiments measured at the same time after injection. The broken axes mark the centroid of each tracer cloud. The ensemble mean tracer cloud represents the average of a large number of identical experiments. The solid coordinate axes move at the ensemble mean velocity

centroids of the individual tracer clouds. These broken axes move at the mean velocity of the individual tracer clouds. If we conducted a large number of identical experiments and averaged the results we would describe the ensemble mean tracer cloud. The solid axes in Figure 1.5 are drawn through the centroid of the ensemble mean tracer cloud. It is clear that the dimensions of the ensemble average tracer cloud are larger than the dimensions of the individual tracer clouds. The corollary is that the ensemble average concentration is smaller than the average within an individual tracer cloud. The biggest differences occur in the early stages of diffusion when the tracer cloud is small relative to the size of the turbulent eddies and once the size of the tracer cloud exceeds the largest turbulent eddies then differences between the averages decrease. In Figure 1.5 the solid axes move at the ensemble mean velocity (i.e. the average velocity that would be determined from a large number of realisations of the flow). It is usual to quantify the rate of diffusion from the ensemble-averaged dimensions of the tracer cloud for the reasons discussed in Section 1.6.2.

### 1.6.5 Measures of Spread

When studying diffusion we are ultimately aiming to predict tracer concentrations. A convenient way to approach this problem is to examine the distances separating tracer particles. If tracer mass $M$ is spread over a distance $L$ in each coordinate direction, then the principle of mass conservation tells us that the average concentration is $M/L^3$. An important measure of the size of a tracer cloud is its variance, which is the mean square displacement about the centre of mass. Equation 1.14 defines the variance of an individual tracer cloud about its centroid. In turbulence modelling we are concerned not with individual tracer clouds but with ensemble averages. The variance of an ensemble of dye clouds about the centre of mass of the ensemble is

$$\Sigma_x^2 = \frac{1}{M} \iiint (x - \langle \bar{x} \rangle)^2 \langle c(x,y,z) \rangle \, dx \, dy \, dz \tag{1.19a}$$

where $\langle \bar{x} \rangle$ = centre of mass of the ensemble given by

$$\langle \bar{x} \rangle = \frac{1}{M} \iiint x \langle c(x,y,z) \rangle \, dx \, dy \, dz \tag{1.19b}$$

From Figure 1.5 it is clear that the ensemble average tracer cloud covers a larger area than either of the individual tracer clouds. Thus the ensemble average variance $\Sigma_x^2$ is larger than the variance of an individual cloud $\sigma_x^2$. Indeed it can be shown (Fischer *et al* 1979) that

$$\Sigma_x^2 = \underbrace{\langle \sigma_x^2 \rangle}_{[1]} + \underbrace{\langle (x - \langle x \rangle)^2 \rangle}_{[2]} \tag{1.20}$$



This equation states that the variance of the ensemble mean tracer cloud is the sum of the mean variance of each tracer cloud about its centre of mass (term [1]) plus the mean square displacement of the centroid of each cloud from the centroid of the ensemble (term [2]). Taylor (1921) investigated diffusion in stationary homogeneous turbulence and developed a theory for predicting the rate of change of $\Sigma_x^2$. Most mixing models are based on Taylor's analysis and we make extensive use of his results in this book. [There is also a body of theory dealing with diffusion of individual tracer clouds relative to their individual centres of mass. This is termed relative diffusion and is described by Csanady (1973), Fisher *et al* (1979) and others. We shall not discuss or use relative diffusion theory in this book.] It is important to note that the ensemble mean concentrations may be lower than the mean concentration observed within an individual tracer cloud. The methods described in Section 1.6.3 can be used to account for these differences.

## 1.7 SOLUTE TRANSPORT IN TURBULENT FLOW

In Section 1.5.3 we outlined the derivation of the three-dimensional advection–diffusion equation in laminar flow

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} + u_z \frac{\partial c}{\partial z} = e_m \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) \tag{1.12}$$

In this section we discuss the application of equation (1.12) to solute transport in stationary turbulent flow.

### 1.7.1 Reynolds Averaging

The famous physicist Osborne Reynolds assumed that in turbulent flow the observed velocities and concentrations can be written

$$u_x = \langle u_x \rangle + u_x' \tag{1.21a}$$

$$u_y = \langle u_y \rangle + u_y' \tag{1.21b}$$

$$u_z = \langle u_z \rangle + u_z' \tag{1.21c}$$

$$c = \langle c \rangle + c' \tag{1.21d}$$

where $u_x$, $u_y$, $u_z$ and $c$ = observed velocities and concentration; $\langle\ \rangle$ denotes an ensemble average; and the prime denotes a deviation from the ensemble average. By definition the average values of $c'$, $u_x'$, $u_y'$ and $u_z'$ are zero. If

equations (1.21) are substituted into equation (1.12) we obtain

$$\frac{\partial}{\partial t}(\langle c\rangle + c') + (\langle u_x\rangle + u_x')\frac{\partial}{\partial x}(\langle c\rangle + c') + (\langle u_y\rangle + u_y')\frac{\partial}{\partial y}(\langle c\rangle + c')$$

$$+ (\langle u_z\rangle + u_z')\frac{\partial}{\partial z}(\langle c\rangle + c')$$

$$= e_m\left[\frac{\partial^2}{\partial x^2}(\langle c\rangle + c') + \frac{\partial^2}{\partial y^2}(\langle c\rangle + c') + \frac{\partial^2}{\partial z^2}(\langle c\rangle + c')\right] \quad (1.22)$$

We then take the ensemble average of equation (1.22) and note that the ensemble average of terms such as

$$\frac{\partial c'}{\partial t},\ u_x'\frac{\partial\langle c\rangle}{\partial x},\ \langle u_x\rangle\frac{\partial c'}{\partial x},\ \frac{\partial^2 c'}{\partial x^2} \quad (1.23)$$

all vanish. The left-hand side of equation (1.22) contains the term

$$u_x'\frac{\partial c'}{\partial x} + u_y'\frac{\partial c'}{\partial y} + u_z'\frac{\partial c'}{\partial z} \quad (1.24)$$

This can be replaced by

$$\frac{\partial}{\partial x}(u_x'c') + \frac{\partial}{\partial y}(u_y'c') + \frac{\partial}{\partial z}(u_z'c') \quad (1.25)$$

because the flow is stationary incompressible continuity requires that

$$\frac{\partial u_x'}{\partial x} + \frac{\partial u_y'}{\partial y} + \frac{\partial u_z'}{\partial z} = 0 \quad (1.26)$$

Making these simplifications we are left with

[1] [2]

$$\frac{\partial\langle c\rangle}{\partial t} + \left(\langle u_x\rangle\frac{\partial\langle c\rangle}{\partial x} + \langle u_y\rangle\frac{\partial\langle c\rangle}{\partial y} + \langle u_z\rangle\frac{\partial\langle c\rangle}{\partial z}\right)$$

[3] [4]

$$= e_m\left(\frac{\partial^2\langle c\rangle}{\partial x^2} + \frac{\partial^2\langle c\rangle}{\partial y^2} + \frac{\partial^2\langle c\rangle}{\partial z^2}\right) - \left(\frac{\partial}{\partial x}\langle u_x'c'\rangle + \frac{\partial}{\partial y}\langle u_y'c'\rangle + \frac{\partial}{\partial z}\langle u_z'c'\rangle\right) \quad (1.27)$$

Term [1] in equation (1.27) is the time rate of change of the ensemble mean concentration. The three terms [2] quantify advection of the ensemble mean concentration by the mean velocity. The three terms [3] quantify molecular diffusion. Comparing equation (1.12) with (1.27) we see that the first seven terms are similar in form except that $\langle u_x\rangle$, $\langle u_y\rangle$, $\langle u_z\rangle$ and $\langle c\rangle$ appear in place



of $u_x$, $u_y$, $u_z$ and $c$. The major difference is the appearance on the right-hand side of equations (1.27) of the three new turbulent transport terms [4].

$$-\frac{\partial}{\partial x}\langle u_x'c'\rangle - \frac{\partial}{\partial y}\langle u_y'c'\rangle - \frac{\partial}{\partial z}\langle u_z'c'\rangle \quad (1.28)$$

Although the ensemble means of the velocity and concentration deviations are zero, the ensemble means of their products are in general non-zero. In other words the fluctuations of velocity and concentration are correlated in some manner (which we have yet to define) and this correlation gives rise to additional transport. The additional transport is termed turbulent diffusion or eddy diffusion.

In rivers eddy diffusion causes tracer to spread far more rapidly than could happen if spreading only occurred as a result of molecular diffusion. This does not mean that molecular diffusion plays no part in turbulent mixing. Molecular diffusion is still the fundamental process which causes tracer to diffuse from a region of high concentration to a region of low concentration. Turbulence has the effect of tearing apart tracer clouds thereby increasing local concentration gradients and accelerating molecular diffusion. It is the combination of turbulent eddies and molecular diffusion which causes rapid tracer mixing in turbulent flow.

In situations where the ergodic hypothesis is valid then time averages can be used in place of ensemble averages. Equation (1.27) then becomes

$$\frac{\partial\bar{c}}{\partial t} + \left(\bar{u}_x\frac{\partial\bar{c}}{\partial x} + \bar{u}_y\frac{\partial\bar{c}}{\partial y} + \bar{u}_z\frac{\partial\bar{c}}{\partial z}\right)$$

$$= e_m\left(\frac{\partial^2\bar{c}}{\partial x^2} + \frac{\partial^2\bar{c}}{\partial y^2} + \frac{\partial^2\bar{c}}{\partial z^2}\right) - \frac{\partial}{\partial x}(\overline{u_x'c'}) - \frac{\partial}{\partial y}(\overline{u_y'c'}) - \frac{\partial}{\partial z}(\overline{u_z'c'}) \quad (1.29)$$

where the overbar denotes a time average. It may appear strange that we have time-averaged and yet produced equation (1.29) which predicts the rate of change of concentration with time. The key to successfully applying the Reynolds decomposition is that the averaging period $T$ used is large compared with the time-scale of the turbulent fluctuations but is small compared with the time-scale of the change in concentration which we wish to study. We have problems using the ergodic hypothesis if the time-scales of the turbulent fluctuations are comparable with the time-scales of the concentration change we are interested in, as can occur in estuarine flows, but we rarely encounter this problem in rivers.

### 1.7.2 The Problem of Closure

In deriving equations (1.27) and (1.29) we have not introduced any approximations. Unfortunately the averaging process has created a new problem because

now turbulent transport of the mean concentration is governed by terms involving the unknown velocity and concentration fluctuations. This is the problem of closure which plagues the subject of turbulence modelling. Before we can solve equations (1.27) and (1.29) we need to relate the turbulent fluxes to some property of the average concentration. We shall see in the next section that this can indeed be done and that under certain conditions the turbulent fluxes are proportional to the spatial gradients of the ensemble mean concentration.

### 1.7.3 Taylor's Analysis of Turbulent Diffusion

In 1921 Taylor published a classic paper in which he made a theoretical analysis of the spreading of a cloud of tracer particles released into stationary, homogeneous turbulence (Taylor 1921). This analysis remains today the key to quantifying turbulent diffusion. Taylor adopted a Lagrangian coordinate system travelling at the mean velocity and examined the rate of longitudinal mixing. If a tracer particle is released at the origin then after a time $t$ it will be located at a longitudinal distance $x$ from the moving origin where

$$x(t) = \int_{\tau=0}^{t} u_x'(\tau)\, d\tau \tag{1.30}$$

and $u_x'$ = the turbulent velocity fluctuation about the mean defined in equation (1.21a). If $N$ particles are released then because of the random nature of the turbulent velocity fluctuations they will be at different locations $x_i$ (where $i = 1, 2, 3 \ldots N$) after time $t$. Now it follows from the conservation of mass that the ensemble mean variance of the resulting tracer cloud is identically equal to the ensemble mean square displacement of tracer particles. Thus

$$\Sigma_x^2 = \langle (x - \bar{x})^2 \rangle \tag{1.31a}$$

Because a Lagrangian coordinate system is being used

$$\langle \bar{x} \rangle = 0 \tag{1.31b}$$

and so

$$\Sigma_x^2 = \langle x^2 \rangle \tag{1.31c}$$

In other words the ensemble mean size of the tracer cloud is the same as the ensemble mean variance of the displacement of the individual particles from the origin of the Lagrangian coordinate system.



We can rewrite equation (1.30)

$$x^2(t) = \left[ \int_{t_1=0}^{t} u_x'(t_1)\, dt_1 \right] \left[ \int_{t_2=0}^{t} u_x'(t_2)\, dt_2 \right]$$

$$= \int_{t_1=0}^{t} \int_{t_2=0}^{t} u_x'(t_1) u_x'(t_2)\, dt_1\, dt_2 \tag{1.32}$$

In equation (1.32) the product

$$u_x'(t_1) u_x'(t_2) \tag{1.33}$$

is the autocovariance between the velocity of a particle at time $t_1$ and the velocity of the same particle at a later time $t_2$. Taking the ensemble average of equation (1.32)

$$\langle x^2 \rangle = \int_{t_1=0}^{t} \int_{t_2=0}^{t} \langle u_x'(t_1) u_x'(t_2) \rangle\, dt_1\, dt_2 \tag{1.34}$$

The Lagrangian autocorrelation function is defined

$$R_x(t_2 - t_1) = \frac{\langle u_x'(t_1) u_x'(t_2) \rangle}{\langle u_x'^2 \rangle} \tag{1.35}$$

where $\sqrt{\langle u_x'^2 \rangle}$ = intensity of turbulence.

Because the turbulence is assumed to be stationary, the autocorrelation can only depend on the time difference $t_2 - t_1$. Combining equations (1.34) and (1.35)

$$\langle x^2 \rangle = \langle u_x'^2 \rangle \int_{t_1=0}^{t} \int_{t_2=0}^{t} R_x(t_2 - t_1)\, dt_2\, dt_1 \tag{1.36}$$

which can be rewritten

$$\langle x^2 \rangle = 2 \langle u_x'^2 \rangle \int_{s=0}^{t} (t - s) R_x(s)\, ds \tag{1.37}$$

The Lagrangian autocorrelation function cannot be predicted theoretically but its general form is known (Figure 1.6). By definition

$$R_x(0) = 1 \tag{1.38}$$

and at large separations the velocity fluctuations become independent so that

$$R_x(\infty) = 0 \tag{1.39}$$

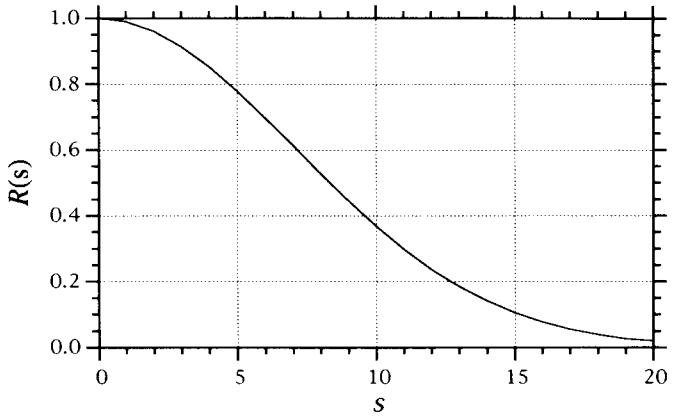


**Figure 1.6.** Schematic representation of the Lagrangian autocorrelation function typical of turbulent flow

Two limiting cases can now be examined. At very small values of $t$, $R_x = 1$ and equation (1.37) becomes

$$\langle x^2 \rangle = \langle u_x'^2 \rangle t^2 \tag{1.40}$$

Thus at small times, the variance increases at a rate proportional to $t^2$. At very large $t$, $R_x = 0$ and equation (1.37) becomes

$$\langle x^2 \rangle \rightarrow 2\langle u_x'^2 \rangle T_x t + \text{constant} \tag{1.41}$$

where

$$T_x = \int_{s=0}^{\infty} R_x(s)\, ds \tag{1.42}$$

$T_x$ is known as the integral Lagrangian time-scale and is a measure of the time taken for a particle to forget its original velocity. The relationship in (1.41) is a very important result which implies that some time after the tracer has been released into turbulent flow the variance of the tracer cloud increases linearly with time. The Lagrangian time-scale tells us how long we have to wait before this happens. Equation (1.41) also states that the rate at which the variance of the tracer cloud grows increases with the square of the turbulent intensity (i.e. spreading occurs more rapidly in highly turbulent flows).

### 1.7.4 Fickian Model of Turbulent Diffusion

In Section 1.5.4 it was shown that one of the features of the Fickian diffusion model is that the variance of a tracer cloud increases linearly with time. Taylor's analysis demonstrates that in stationary homogeneous turbulence the



variance of a tracer cloud increases linearly with time at asymptotically large times. This suggests that in stationary homogeneous turbulence turbulent diffusion can also be modelled using Fick's law provided sufficient time has elapsed since tracer injection. By analogy with equation (1.12) we can write

$$\frac{\partial \langle c \rangle}{\partial t} + \langle u_x \rangle \frac{\partial \langle c \rangle}{\partial x} + \langle u_y \rangle \frac{\partial \langle c \rangle}{\partial y} + \langle u_z \rangle \frac{\partial \langle c \rangle}{\partial z}$$

$$= (e_m + e_t)\left(\frac{\partial^2 \langle c \rangle}{\partial x^2} + \frac{\partial^2 \langle c \rangle}{\partial y^2} + \frac{\partial^2 \langle c \rangle}{\partial z^2}\right) \tag{1.43}$$

where $e_t$ = turbulent diffusion coefficient (eddy diffusivity) which is assumed to be isotropic and homogeneous. The effects of turbulent dispersion and molecular diffusion are assumed to be additive although this is largely academic because invariably $e_t \gg e_m$. Comparing equations (1.43) and (1.29)

$$e_t \frac{\partial^2 \langle c \rangle}{\partial x^2} = -\frac{\partial}{\partial x} \langle u_x' c' \rangle \tag{1.44a}$$

$$e_t \frac{\partial^2 \langle c \rangle}{\partial y^2} = -\frac{\partial}{\partial y} \langle u_y' c' \rangle \tag{1.44b}$$

$$e_t \frac{\partial^2 \langle c \rangle}{\partial z^2} = -\frac{\partial}{\partial z} \langle u_z' c' \rangle \tag{1.44c}$$

By analogy with equation (1.2) the turbulent diffusive fluxes are

$$J_x = \langle u_x' c' \rangle = -e_t \frac{\partial \langle c \rangle}{\partial x} \tag{1.45a}$$

$$J_y = \langle u_y' c' \rangle = -e_t \frac{\partial \langle c \rangle}{\partial y} \tag{1.45b}$$

$$J_z = \langle u_z' c' \rangle = -e_t \frac{\partial \langle c \rangle}{\partial z} \tag{1.45c}$$

Equations (1.43)–(1.45) constitute the Fickian model for turbulent diffusion.

There is some conjecture about the validity of this model. In his analysis, Taylor (1921) stopped after showing that at asymptotically large times the variance of the tracer cloud increases linearly with time and he did not go so far as to postulate equation (1.43). Fischer *et al* (1979) argue that the linear growth of variance is a necessary condition for the validity of equation (1.43) but that it is not a sufficient condition which establishes beyond doubt the validity of the Fickian model for turbulent diffusion. Despite these theoretical shortcomings the model is widely used to address practical mixing problems and there is a large body of empirical evidence to support its use in river mixing problems. There are difficulties applying the model in the near-field when

buoyancy and momentum are important (Tennekes and Lumley 1972) but these stem from the fact that the turbulence is neither homogeneous nor stationary. In the mid and far-fields the Fickian model provides an economical and tolerably accurate model for turbulent diffusion. Fickian models are used extensively in practical mixing problems when detailed measurements of turbulence are not available and hence the use of more sophisticated models is not feasible.

The Fickian model of turbulent diffusion ensures closure of the turbulence equations provided we can estimate the value of the eddy diffusivity. The eddy diffusivity $e_t$ is in some ways analogous to the molecular diffusivity $e_m$ but there are three important differences. Firstly, as a rough guide, $e_t$ is of the order $10^{-3}\ m^2 s^{-1}$ whereas $e_m$ is typically $10^{-9}\ m^2 s^{-1}$. Secondly, whereas the molecular diffusivity is a property of the fluid, the eddy diffusivity is a property of the flow and varies with the velocity, turbulence and geometry of the flow. Thirdly, equations (1.43)–(1.45) only hold when $t \gg T_x$ where $T_x$ is the integral Lagrangian time-scale defined in equation (1.42).

### 1.7.5 Eddy Viscosity

Equations (1.45) are far older than Taylor's analysis and it is worthwhile making a digression to discuss momentum transfer in turbulent flows. Turbulent eddies transfer both momentum and mass and experimental work shows that in many flows the rates of transfer are almost the same. This assumption is known as the Reynolds analogy and holds provided the tracer is neutrally buoyant and the concentration is low. Consider the case of laminar two-dimensional flow. The velocity is assumed uniform across and along the channel. The no-slip condition requires that the velocity at the bed is zero and this causes a vertical velocity gradient (Figure 1.7). In laminar flow random molecular movement causes a small exchange of flow between adjacent fluid layers whereas in turbulent flow a much larger exchange is effected by turbulent eddies. As the mean velocity of the adjacent layers differs, any vertical exchange of fluid causes a transfer of momentum. From Newton's second law this results in a shear stress. In plane laminar shear flow the viscous shear stress is

$$\tau = \rho \nu \frac{\partial u_x}{\partial y} \tag{1.46}$$

where $\nu = \mu/\rho$ = kinematic viscosity of the fluid, $u_x(y)$ = velocity parallel to the bed, $y$ = height above the bed and $\tau$ = viscous shear stress. In turbulent flow eddies transfer momentum vertically more rapidly than in laminar flow and the shear stresses are markedly higher. The shear stresses which arise from turbulent velocity fluctuations are called turbulent or Reynolds stresses. By applying the Reynolds decomposition to the equations of motion (see, for



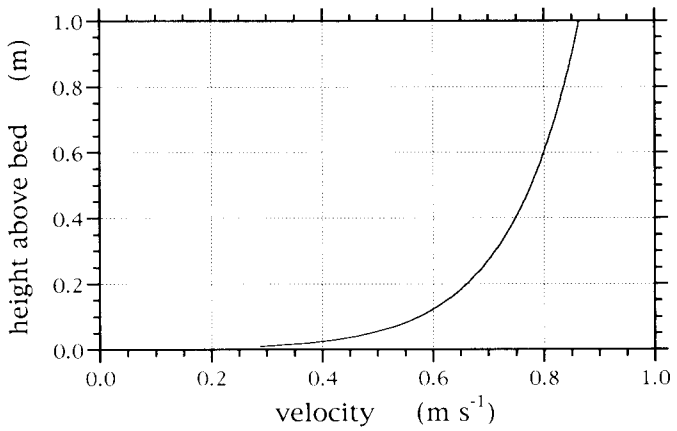


**Figure 1.7.** Sketch showing the logarithmic velocity profile predicted from equation (1.13) with $u^* = 0.05\ m\ s^{-1}$, $y_0 = 0.001$ m and $h = 1.0$ m

example, Hinze 1972 or Schlichting 1968) the Reynolds stress is found to be

$$\tau_t = \rho \langle u_x' u_y' \rangle \tag{1.47}$$

where $u_x'$ and $u_y'$ = turbulent velocity fluctuations in the $x$ and $y$ directions and $\tau_t$ = turbulent shear stress (the Reynolds stress). Note that equation (1.47) for the Reynolds stress contains the term

$$\langle u_x' u_y' \rangle \tag{1.48}$$

and is similar to equations (1.45) for the eddy diffusivities which contain terms such as

$$\langle u_x' c' \rangle \tag{1.49}$$

The Reynolds stresses arise from correlations between turbulent velocity fluctuations whereas eddy diffusivities arise from correlations between turbulent velocity and concentration fluctuations. When the Reynolds stresses appear in the momentum equations they introduce the problem of closure discussed earlier. In 1877 Boussinesq suggested that by analogy with equation (1.46) the Reynolds stresses can be related to the gradient in the mean velocity. Thus

$$\tau_t = \rho \nu_t \frac{\partial \langle u_x \rangle}{\partial y} \tag{1.50}$$

where $\langle u_x \rangle$ = ensemble mean velocity and $\nu_t$ = the eddy viscosity. Whereas the kinematic viscosity $\nu$ is a property of the fluid, the eddy viscosity $\nu_t$ is a property of the flow and depends on the intensity of turbulence and the length scales of the dominant eddies.

Reynolds pointed out that the turbulent eddies which transport momentum

also transport mass. Equation (1.50) states that the rate of transfer of momentum is proportional to the velocity gradient and by analogy the rate of transfer of mass is proportional to the concentration gradient. This analogy leads to

$$J_y = -e_t \frac{\partial \langle c \rangle}{\partial y} \tag{1.51}$$

where $J_y$ = vertical flux of tracer mass; $\langle c \rangle$ = ensemble mean concentration and $e_t$ = vertical turbulent diffusion coefficient. Note that $e_t$ and $\nu_t$ have the same units (i.e. $m^2 s^{-1}$). Reynolds analogy is that for tracer mass

$$e_t = \nu_t \tag{1.52}$$

In some situations (e.g. when dealing with buoyant tracers or high sediment concentrations) Reynolds analogy breaks down and it is necessary to write

$$e_t = Sc\ \nu_t \tag{1.53}$$

where $Sc$ = turbulent Schmidt number, which must be determined experimentally. Typically in rivers $0.3 < Sc < 1.0$.

### 1.7.6 Prandtl's Mixing Length Hypothesis

In 1925 Prandtl published a theory for predicting the eddy viscosity in turbulent shear flow (see, for example, Hinze 1972) which has become known as the mixing length hypothesis. This hypothesis is described in more detail in Section 2.2 but the two main results are stated here. Prandtl noted that the kinetic theory of gases gives rise to an expression for viscosity

$$\nu_g = \tfrac{1}{3} L_g u_m \tag{1.54}$$

where $\nu_g$ = kinematic viscosity of the gas, $L_g$ = mean free path of the gas molecules (which is a measure of the distance molecules travel between collisions) and $u_m$ = a typical molecular velocity. He drew an analogy between the kinetic theory of gases and the behaviour of eddies in turbulent flow and suggested that

$$\nu_t = u_t L_m \tag{1.55}$$

where $\nu_t$ = eddy viscosity, $L_m$ = Prandtl's mixing length and $u_t$ = a turbulent velocity.

Prandtl's theory has two deficiencies. Firstly, it is by no means certain that eddies interact in the same way as gas molecules and, secondly, there is a large range of eddy sizes in turbulent flow so that it is not always easy to decide on a value for $L_m$. Nevertheless equation (1.55) is a very useful equation which indicates that the eddy viscosity depends on a turbulent length scale and a turbulent velocity. Indeed, this equation can be derived purely on dimensional



grounds. We may choose not to accept Prandtl's mixing length hypothesis in its entirety but can still retain equation (1.55) and estimate $u_t$ and $L_m$ experimentally. For example, equation (1.41) indicates that some time after its release, the variance of a tracer cloud increases linearly with time. Equations (1.52) and (1.55) imply the existence of a constant eddy diffusivity which can be evaluated from equation (1.41) to be

$$e_t = \frac{1}{2} \frac{d}{dt} \langle x^2 \rangle \tag{1.56}$$

Comparing equations (1.56) and (1.41)

$$e_t = \langle u_x'^2 \rangle T_x \tag{1.57}$$

where $T_x$ is the integral Lagrangian time-scale defined by equation (1.42). We can define an integral length scale

$$L_x = \sqrt{\langle u_x'^2 \rangle}\, T_x \tag{1.58}$$

Substituting equation (1.58) into (1.57)

$$e_t = \sqrt{\langle u_x'^2 \rangle}\, L_x \tag{1.59}$$

There is a close similarity between equations (1.55) and (1.59) and they are identical if

$$e_t = \nu_t \tag{1.60a}$$

$$L_x = L_m \tag{1.60b}$$

$$u_t = \sqrt{\langle u_x'^2 \rangle} \tag{1.60c}$$

The time and length scales $T_x$, $L_x$ and $L_m$ cannot be determined theoretically but they can be estimated experimentally by examining the correlation structure of the turbulent velocity and concentration fluctuations or by determining the point at which tracer clouds begin to increase linearly.

In the second part of his analysis, Prandtl postulated values for $u_t$ and $L_m$ and suggested that

$$\nu_t = y^2 \frac{\partial \langle u_x \rangle}{\partial y} \tag{1.61}$$

where $y$ = height above the bed and $\langle u_x \rangle$ = ensemble mean longitudinal velocity. Equation (1.61) has been validated by experiments for boundary layer flows but does not hold universally. For example, at the centre of a pipe the radial velocity gradient is zero and equation (1.61) implies that the eddy viscosity (and hence the eddy diffusivity) is zero. Experimental results show, however, that the eddy diffusivity at the centre of the pipe is about 80% of the cross sectional averaged value (Rodi 1980). In rivers turbulence is mostly generated by friction at the bed and the mixing length hypothesis can be

applied with some confidence. In Chapter 2 it is used to derive an equation for the vertical eddy diffusivity.

### 1.7.7 Diffusion in Non-homogeneous Turbulence

In Section 1.7.4 it was argued that equation (1.43) provides a reasonable description of turbulent diffusion in stationary homogeneous turbulence. The turbulent diffusion coefficient $e_t$ was assumed to be homogeneous (i.e. not to vary spatially) and isotropic (i.e. the same in all three coordinate directions) in line with Taylor's analysis. Because the diffusion coefficient is isotropic, the diffusive flux in any direction is the product of the concentration gradient in that direction and the diffusion coefficient. One ramification is that if we rotate the coordinate axes then the diffusion equation is the same in the new coordinate system as in the old. It is left as an exercise for the reader to confirm this. The molecular diffusion coefficient is a property of the fluid and so is homogeneous and isotropic. The turbulent diffusion coefficient, however, is a property of the flow and consequently may vary spatially and with direction. For example, in river channels where the ratio of width/depth is large there are good reasons for expecting the transverse diffusivity to be larger than the vertical diffusivity. We will make an intuitive extension of equation (1.43) to the non-isotropic case and then discuss its subtleties.

In a straight channel the $x$ axis is aligned with the main flow, the $y$ axis is taken as vertical and the $z$ axis is taken as transverse across the channel. The flow is assumed stationary and the turbulent diffusion coefficient non-isotropic and non-uniform. It seems logical to rewrite equation (1.43) as

$$\frac{\partial c}{\partial t}+\frac{\partial}{\partial x}(u_x c)+\frac{\partial}{\partial y}(u_y c)+\frac{\partial}{\partial z}(u_z c)=\frac{\partial}{\partial x}\left(e_x\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(e_y\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(e_z\frac{\partial c}{\partial z}\right) \tag{1.62}$$

Equation (1.62) is written in the so-called conservation form. Continuity requires that for incompressible flow

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0 \tag{1.63}$$

Expanding the spatial derivatives on the left-hand side of equation (1.62) and using equation (1.63) yields the advection form

$$\frac{\partial c}{\partial t}+u_x\frac{\partial c}{\partial x}+u_y\frac{\partial c}{\partial y}+u_z\frac{\partial c}{\partial z}=\frac{\partial}{\partial x}\left(e_x\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(e_y\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(e_z\frac{\partial c}{\partial z}\right) \tag{1.64}$$

If the diffusion coefficients are constant then equation (1.64) becomes

$$\frac{\partial c}{\partial t}+u_x\frac{\partial c}{\partial x}+u_y\frac{\partial c}{\partial y}+u_z\frac{\partial c}{\partial z}=e_x\frac{\partial^2 c}{\partial x^2}+e_y\frac{\partial^2 c}{\partial y^2}+e_z\frac{\partial^2 c}{\partial z^2} \tag{1.65}$$



Equations (1.62)–(1.64) are the basis for most of the mixing calculations in this book and are of fundamental importance. The molecular diffusivity is assumed to be negligibly small compared with the turbulent diffusivity and is omitted.

One of the subtleties of equations (1.62), (1.64) and (1.65) appears if the coordinate axes are rotated to new coordinate axes $\alpha(x,y,z)$, $\beta(x,y,z)$ and $\gamma(x,y,z)$. To make this transformation the chain rule of differentials is used

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial\alpha}\frac{\partial\alpha}{\partial x}+\frac{\partial}{\partial\beta}\frac{\partial\beta}{\partial x}+\frac{\partial}{\partial\gamma}\frac{\partial\gamma}{\partial x} \tag{1.66a}$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial\alpha}\frac{\partial\alpha}{\partial y}+\frac{\partial}{\partial\beta}\frac{\partial\beta}{\partial y}+\frac{\partial}{\partial\gamma}\frac{\partial\gamma}{\partial y} \tag{1.66b}$$

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial\alpha}\frac{\partial\alpha}{\partial z}+\frac{\partial}{\partial\beta}\frac{\partial\beta}{\partial z}+\frac{\partial}{\partial\gamma}\frac{\partial\gamma}{\partial z} \tag{1.66c}$$

It is left as an exercise for the reader to show that equation (1.64) becomes

$$\frac{\partial c}{\partial t}+u_\alpha\frac{\partial c}{\partial\alpha}+u_\beta\frac{\partial c}{\partial\beta}+u_\gamma\frac{\partial c}{\partial\gamma}$$
$$=e_{\alpha\alpha}\frac{\partial^2 c}{\partial\alpha^2}+e_{\beta\beta}\frac{\partial^2 c}{\partial\beta^2}+e_{\gamma\gamma}\frac{\partial^2 c}{\partial\gamma^2}+e_{\alpha\beta}\frac{\partial^2 c}{\partial\alpha\partial\beta}+e_{\alpha\gamma}\frac{\partial^2 c}{\partial\alpha\partial\gamma}+e_{\beta\gamma}\frac{\partial^2 c}{\partial\beta\partial\gamma} \tag{1.67}$$

where $u_\alpha$, $u_\beta$ and $u_\gamma$ are velocities in the new coordinate directions (which are functions of $u_x$, $u_y$, $u_z$, $\alpha$, $\beta$ and $\gamma$) while $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\gamma\gamma}$, $e_{\alpha\beta}$, $e_{\alpha\gamma}$ and $e_{\beta\gamma}$ are diffusion coefficients in the new coordinate directions (which are functions of $e_x$, $e_y$, $e_z$, $\alpha$, $\beta$ and $\gamma$). The important thing to note about equation (1.67) is the appearance of the three cross-derivative dispersion terms at the end. These arise because turbulent diffusion cannot in general be described by the vector $(e_x, e_y, e_z)$ but requires at least a second-order tensor (Dagan 1969). In equations (1.62), (1.64) and (1.65) there are no cross-derivative terms because the coordinate axes happen to coincide with the principal axes of the diffusion tensor.

An obvious question is whether we can always arrange for the coordinate axes and the principal axes of the diffusion tensor to coincide. Unfortunately in natural channels this is not possible, particularly near the banks. In a straight, infinitely wide, rectangular channel the flow is two-dimensional, the isovels are everywhere horizontal and the only velocity shear occurs in the vertical direction. Consequently one principal axis of the diffusion tensor is everywhere vertical. There is no difficulty in this situation aligning the coordinate axes with the principal axes of the diffusion tensor. In an irregular channel, or a channel of finite width where friction at the banks is important, the isovels are no longer horizontal (see Figure 1.8). Prandtl's mixing length hypothesis furnishes estimates of the eddy diffusivity along axes normal to the isovels but

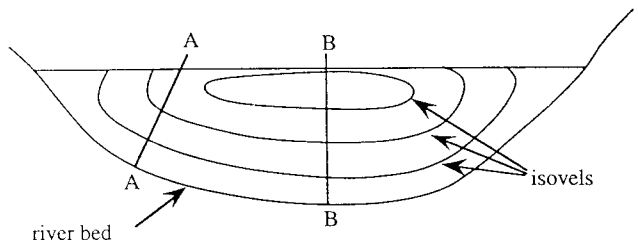


**Figure 1.8.** Diagram showing the isovels in a natural channel. Prandtl's mixing length hypothesis furnishes estimates of the eddy diffusivity along axes (such as A–A and B–B) which are orthogonal to the isovels

this direction is no longer fixed. Strictly speaking the cross-product terms in equation (1.67) should be retained, especially near the banks. This greatly complicates the solution of the mixing equation. In most practical problems, therefore, the cross-products are simply neglected (Fischer *et al* 1979, Chatwin and Allen 1985). This does not appear to cause serious problems when modelling tracer concentrations in rivers and estuaries. The eddy diffusivities $e_x$, $e_y$ and $e_z$ are commonly estimated by fitting solutions to measured tracer concentrations and the calibration process is able to compensate for any slight deficiencies in the model structure. It should be remembered, however, that near the banks and in regions where there are secondary currents equations (1.62), (1.64) and (1.65) provide only an approximate description of tracer mixing.

## 1.8 SIMPLIFYING THE MIXING EQUATIONS

### 1.8.1 Introduction

The three-dimensional advection diffusion equation (1.64) forms the basis for analysing mixing problems in rivers. One practical difficulty is that the full equation requires more information on water depths, velocities and diffusion coefficients than can conveniently be gathered during field experiments in natural channels. In addition the three-dimensional equation is difficult to solve in a complex natural channel. In particular mixing problems, however, some of the terms in equation (1.64) are negligibly small and the problem can be simplified. Figure 1.9 and Table 1.2 summarise a number of common river mixing problems.

Firstly, consider mixing downstream from a point source. It is convenient to define three mixing zones: the near-field, mid-field and far-field. Table 1.2 gives approximate formulae for estimating the length of each mixing zone as a guide to the reader: further details of how these formulae were derived are given in Chapter 2 (near-field) and Chapter 3 (mid-field). Table 1.2 also indicates the dimensionality of the mixing equations in each problem. In the



Table 1.2. A summary of important river mixing problems and the terms required to model them

| Type of Source | | | Type of Solution (b) | Terms required (a) | | Dimensions |
|---|---|---|---|---|---|---|
| | | | | Advection | Dispersion | |
| Time | Space (a) | | | | | |
| instantaneous | point source | near-field | $0 < x < u_x h^2/2e_y$ | $x$ (c) | $x, y, z$ | 3 |
| | | mid-field | $u_x h^2/2e_y < x < v_x b^2/2k_z$ | $x$ | $x, z$ | 2 |
| | | far-field | $v_x b^2/2k_z < x$ | $x$ | $x$ | 1 |
| steady | point source | near-field | $0 < x < u_x h^2/2e_y$ | $x$ (c) | $y, z$ | 3 (e) |
| | | mid-field | $u_x h^2/2e_y < x < v_x b^2/2k_z$ | $x$ | $z$ | 2 (e) |
| | | far-field | $v_x b^2/2k_z < x$ | — (d) | — (d) | 0 (d) |
| instantaneous | $z$-line source | near-field | $0 < x < u_x h^2/2e_y$ | $x$ | $x, y$ | 2 |
| | | far-field | $v_x b^2/2k_z < x$ | $x$ | $x$ | 1 |
| steady | $z$-line source | near-field | $0 < x < u_x h^2/2e_y$ | $x$ | $y$ | 2 (e) |
| | | far-field | $v_x b^2/2k_z < x$ | — (d) | — (d) | 0 (d) |
| instantaneous | $y$-line source | mid-field | $0 < x < v_x b^2/2k_z$ | $x$ | $x, z$ | 2 |
| | | far-field | $v_x b^2/2k_z < x$ | $x$ | $x$ | 1 |
| steady | $y$-line source | mid-field | $0 < x < v_x b^2/2k_z$ | $x$ | $z$ | 2 (e) |
| | | far-field | $v_x b^2/2k_z < x$ | — (d) | — (d) | 0 (d) |

Notes: (a) $x$ = longitudinal, $y$ = vertical, $z$ = transverse. (b) $e_y$ = vertical diffusion coefficient, $u_x$ = longitudinal velocity, $v_x$ = depth-averaged velocity, $k_z$ = transverse dispersion coefficient. (c) On a small scale $y$, $z$ advection may be present in the prototype. (d) Concentration is constant (fully-mixed). (e) The dimensionality can be reduced by one if the coordinate system used travels downstream at mean velocity. Tracer is assumed to be conservative.

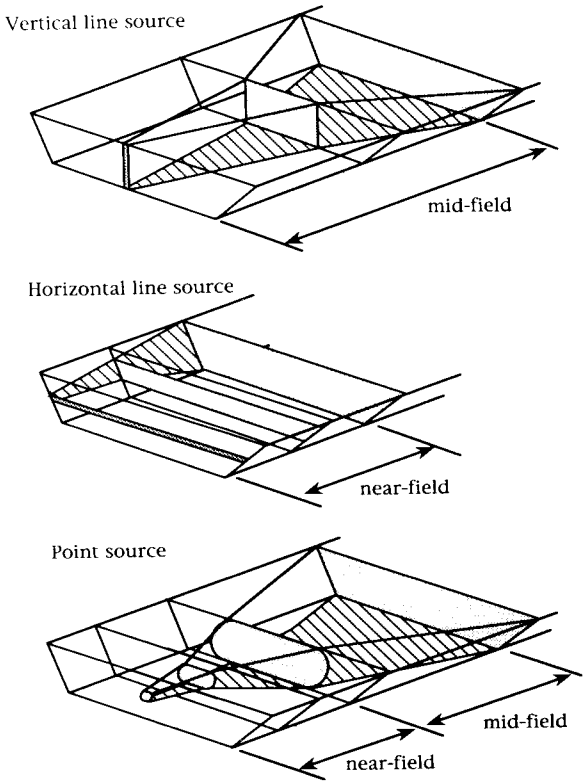


**Figure 1.9.** Diagram showing the general characteristics of tracer plumes downstream from three different types of tracer source. Below a point source both vertical and transverse concentration gradients are significant in the near-field, transverse concentration gradients are significant in the mid-field and tracer is well-mixed across the channel in the far-field. If the source is steady then longitudinal concentration gradients are small. Tracer is vertically well-mixed everywhere downstream from a vertical line source and transversely well-mixed everywhere downstream from a transverse line source. In practice momentum and buoyancy may affect tracer mixing in the near-field

near-field, tracer released from an instantaneous point source diffuses vertically, transversely and longitudinally as it is advected downstream and the full three-dimensional mixing equation is required. (In the near-field buoyancy and momentum may also be important depending on the nature of the source.) In the near-field it is possible to align the $x$ axis with the flow so that $u_y = u_z = 0$. (It may not be possible, however, to align the $x$ axis with the main flow throughout the mid-field of a natural channel because the thalweg (i.e. the main flow path) tends to meander from one side of the channel to



the other.) In this situation equation (1.64) becomes

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\left(e_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(e_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(e_z \frac{\partial c}{\partial z}\right) \tag{1.68a}$$

If the tracer source is steady then the first term in equation (1.68a) vanishes and the longitudinal concentration gradients become negligibly small so that

$$u_x \frac{\partial c}{\partial x} = \frac{\partial}{\partial y}\left(e_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(e_z \frac{\partial c}{\partial z}\right) \tag{1.68b}$$

If tracer originates from a steady transverse line source then transverse concentration gradients are everywhere small and

$$u_x \frac{\partial c}{\partial x} = \frac{\partial}{\partial y}\left(e_y \frac{\partial c}{\partial y}\right) \tag{1.68c}$$

Equation (1.68c) is used extensively in Chapter 2 for vertical mixing in the near-field.

The mid-field is the region in which vertical concentration gradients are small and attention can be focused on transverse and longitudinal changes in the depth-averaged concentration. The problem can be simplified by averaging equation (1.64) over the depth (as is discussed in Section 1.8.2) to give a two-dimensional model. In the mid-field it is not always possible to align the $x$ axis with the main flow and both longitudinal and transverse advection are important. For an instantaneous source both transverse and longitudinal dispersion are important in the mid-field, but for a steady source the latter can be neglected.

In the far-field transverse concentration gradients become unimportant, attention is focused on the cross-sectional averaged concentration and equation (1.64) is averaged over the cross-section (as is discussed in Section 1.8.6) to yield a one-dimensional model. If the source is steady, the tracer is conservative and there are no tributary inflows, then the tracer concentration in the far-field is constant. If the source is non-steady (e.g. following an instantaneous discharge) then the resulting one-dimensional equation can be used to model advection and longitudinal dispersion.

### 1.8.2 Depth Averaging: Part One

In most rivers the aspect ratio (width/depth) is large and tracer becomes well mixed vertically long before becoming well-mixed transversely. Consequently vertical mixing is only important close to the tracer source. In the mid-field we can neglect vertical concentration gradients and focus our attention on transverse and longitudinal changes of the depth-averaged concentration. This requires only a two-dimensional equation. Some care must be taken when reducing the dimensionality of a mixing problem. If the mixing equations are

simplified incorrectly the resulting model is unlikely to predict the behaviour of tracer accurately and the only way to make the model match field data is by distorting the model coefficients. There are several published examples of mixing equations being incorrectly simplified [despite excellent expositions of the correct procedures given by (Holly 1975) and (Yotsukura 1977)] and as a result some published mixing coefficients are seriously in error. To derive equations governing the depth-averaged tracer concentration it is not permissible simply to neglect all the terms in equation (1.64) which vary with depth. The correct procedure for simplifying the mixing equations is to integrate over the depth term by term taking careful account of any depth variations of velocity and concentration. The derivation is given here in some detail and although the calculus may be a little daunting the reader is urged to persevere in order to appreciate that spatial variations of longitudinal and transverse velocity are very important in mixing tracer.

The three-dimensional advection/diffusion equation in rectangular Cartesian coordinates is

$$\frac{\partial c}{\partial t}+\frac{\partial}{\partial x}(u_x c)+\frac{\partial}{\partial y}(u_y c)+\frac{\partial}{\partial z}(u_z c)=\frac{\partial}{\partial x}\left(e_x\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(e_y\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(e_z\frac{\partial c}{\partial z}\right) \tag{1.62}$$

We integrate equation (1.62) from the bed $y=b(x,z)$ to the surface $y=a(x,z)$ assuming steady flow. Extensive use is made of Leibnitz's rule.

$$\int_{b(x,z)}^{a(x,z)}\frac{\partial}{\partial x}\,\mathrm{f}(x,y)\,\mathrm{d}y=\frac{\partial}{\partial x}\int_{b(x,z)}^{a(x,z)}\mathrm{f}(x,y)\,\mathrm{d}y-\mathrm{f}(x,a)\frac{\partial a}{\partial x}+\mathrm{f}(x,b)\frac{\partial b}{\partial x} \tag{1.69}$$

Integrating equation (1.62) term by term leads to

$$\int_b^a\frac{\partial c}{\partial t}\,\mathrm{d}y+\frac{\partial}{\partial x}\int_b^a u_x c\,\mathrm{d}y-(u_x c)_a\frac{\partial a}{\partial x}+(u_x c)_b\frac{\partial b}{\partial x}+(u_y c)_a-(u_y c)_b$$

$$+\frac{\partial}{\partial z}\int_b^a u_z c\,\mathrm{d}y-(u_z c)_a\frac{\partial a}{\partial z}+(u_z c)_b\frac{\partial b}{\partial z}$$

$$=\frac{\partial}{\partial x}\int_b^a e_x\frac{\partial c}{\partial x}\,\mathrm{d}y-\left(e_x\frac{\partial c}{\partial x}\right)_a\frac{\partial a}{\partial x}+\left(e_x\frac{\partial c}{\partial x}\right)_b\frac{\partial b}{\partial x}+\left(e_y\frac{\partial c}{\partial y}\right)_a-\left(e_y\frac{\partial c}{\partial y}\right)_b$$

$$+\frac{\partial}{\partial z}\int_b^a e_z\frac{\partial c}{\partial z}\,\mathrm{d}y-\left(e_z\frac{\partial c}{\partial z}\right)_a\frac{\partial a}{\partial z}+\left(e_z\frac{\partial c}{\partial z}\right)_b\frac{\partial b}{\partial z} \tag{1.70}$$



Rearranging equation (1.70) gives

$$\int_b^a\frac{\partial c}{\partial t}\,\mathrm{d}y+\frac{\partial}{\partial x}\int_b^a u_x c\,\mathrm{d}y+\frac{\partial}{\partial z}\int_b^a u_z c\,\mathrm{d}y$$

[1]

$$=\frac{\partial}{\partial x}\int_b^a e_x\frac{\partial c}{\partial x}\,\mathrm{d}y+\frac{\partial}{\partial z}\int_b^a e_z\frac{\partial c}{\partial z}\,\mathrm{d}y+\left[c\left(u_x\frac{\partial a}{\partial x}-u_y+u_z\frac{\partial a}{\partial z}\right)\right]_a$$

[2] [3]

$$-\left[c\left(u_x\frac{\partial b}{\partial x}-u_y+u_z\frac{\partial b}{\partial z}\right)\right]_b-\left(e_x\frac{\partial c}{\partial x}\frac{\partial a}{\partial x}-e_y\frac{\partial c}{\partial y}+e_z\frac{\partial c}{\partial z}\frac{\partial a}{\partial z}\right)_a$$

[4]

$$+\left(e_x\frac{\partial c}{\partial x}\frac{\partial b}{\partial x}-e_y\frac{\partial c}{\partial y}+e_z\frac{\partial c}{\partial z}\frac{\partial b}{\partial z}\right)_b \tag{1.71}$$

At both the water surface and the river bed the fluxes of water and tracer normal to the boundaries are zero. We can use these boundary conditions to show that terms [1]–[4] are all identically zero. Some readers may be happy to accept this statement without question and proceed directly to equation (1.76).

The assumption of zero flow normal to the boundary is called the kinematic boundary condition and in steady flow requires (Yih 1969)

$$u_x\frac{\partial\eta}{\partial x}+u_y+u_z\frac{\partial\eta}{\partial z}=0 \tag{1.72}$$

where $\eta(x,z)$ = coordinates of the boundary. In equation (1.71) setting $\eta(x,z)=a(x,z)$ eliminates term [1] and setting $\eta(x,z)=b(x,z)$ eliminates term [2]. The tracer flux normal to both boundaries must also be zero. The normal vectors to the water surface and the bed are, respectively:

$$\boldsymbol{n}_a=\frac{\partial a}{\partial x}\boldsymbol{i}-\boldsymbol{j}+\frac{\partial a}{\partial z}\boldsymbol{k} \tag{1.73a}$$

$$\boldsymbol{n}_b=\frac{\partial b}{\partial x}\boldsymbol{i}-\boldsymbol{j}+\frac{\partial b}{\partial z}\boldsymbol{k} \tag{1.73b}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ = unit normal vectors in the $x$, $y$ and $z$ directions respectively; and $\boldsymbol{n}_a$ and $\boldsymbol{n}_b$ = normal vectors to the water surface and the bed. The tracer flux vector at any point is

$$\boldsymbol{J}=e_x\frac{\partial c}{\partial x}\boldsymbol{i}+e_y\frac{\partial c}{\partial y}\boldsymbol{j}+e_z\frac{\partial c}{\partial z}\boldsymbol{k} \tag{1.74}$$

As the flux normal to a boundary is zero, it follows that

$$\boldsymbol{J}\cdot\boldsymbol{n}_a = \boldsymbol{J}\cdot\boldsymbol{n}_b = 0 \tag{1.75}$$

Substituting equations (1.73) and (1.74) into (1.75) indicates that terms [3] and [4] of equation (1.71) are zero.

The integrals over the depth can be written

$$\frac{\partial}{\partial x}\int_a^b uc\,\mathrm{d}y = \frac{\partial}{\partial x}(h\,\overline{uc}) \tag{1.76}$$

where the overbar denotes a depth average and $h$ = local depth. Substituting equation (1.76) in (1.71) gives

$$h\frac{\partial \bar{c}}{\partial t} + \frac{\partial}{\partial x}(h\,\overline{u_x c}) + \frac{\partial}{\partial z}(h\,\overline{u_z c}) = \frac{\partial}{\partial x}\left(h\,\overline{e_x\frac{\partial c}{\partial x}}\right) + \frac{\partial}{\partial z}\left(h\,\overline{e_z\frac{\partial c}{\partial z}}\right) \tag{1.77}$$

The velocities and concentrations can be expressed as the sum of a depth-average and a vertical deviation. This is another application of Reynolds' decomposition introduced earlier

$$u_x = v_x + u_x' \tag{1.78a}$$

$$u_z = v_z + u_z' \tag{1.78b}$$

$$c = s + c' \tag{1.78c}$$

where

$$v_x = \frac{1}{h}\int_b^a u_x\,\mathrm{d}y \tag{1.79a}$$

$$v_z = \frac{1}{h}\int_b^a u_z\,\mathrm{d}y \tag{1.79b}$$

$$s = \frac{1}{h}\int_b^a c\,\mathrm{d}y \tag{1.79c}$$

By definition the depth average of the deviations $c'$, $u_x'$ and $u_z'$ are zero. The depth-average of a product can be expressed as follows

$$\overline{u_x c} = \overline{(v_x + u_x')(s + c')} = v_x s + \overline{u_x' c'}$$

since

$$\overline{u_x'}\;\;\overline{c'} \quad 0 \tag{1.80}$$



If we assume that $e_x$ and $e_z$ do not vary with depth then

$$\overline{e_x\frac{\partial c}{\partial x}} = e_x\frac{\partial s}{\partial x} \tag{1.81a}$$

$$\overline{e_z\frac{\partial c}{\partial z}} = e_z\frac{\partial s}{\partial z} \tag{1.81b}$$

Making these substitutions into equation (1.77) yields

$$h\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}(hv_x s) + \frac{\partial}{\partial z}(hv_z s) = \frac{\partial}{\partial x}\left(-h\,\overline{u_x' c'} + he_x\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial z}\left(-h\,\overline{u_y' c'} + he_z\frac{\partial s}{\partial z}\right) \tag{1.82}$$

In equation (1.82)

$$v_x s \quad \text{and} \quad v_y s \tag{1.83a}$$

are the advective fluxes whereas

$$e_x\frac{\partial s}{\partial x} \quad \text{and} \quad e_z\frac{\partial s}{\partial z} \tag{1.83b}$$

are the turbulent diffusive fluxes in the longitudinal and transverse directions respectively. The terms

$$-\overline{u_x' c'} \quad \text{and} \quad -\overline{u_z' c'} \tag{1.83c}$$

arise as a result of depth averaging and quantify the additional transport which results from non-uniformities over the depth of velocity and concentration. This additional transport is termed dispersion. A better name is shear dispersion to emphasise that it arises from non-uniformities of velocity. The next section describes the physical processes which give rise to shear dispersion.

### 1.8.3 Dispersion in Turbulent Shear Flow

In uniform flow (where the velocity is everywhere the same) advection moves a tracer cloud bodily downstream without causing any distortion or spreading. Velocity is seldom uniform in a natural channel because of boundary friction. What then is the effect on the tracer cloud of velocity shear?

Figure 1.10 shows a river channel in which there is pronounced vertical velocity shear arising from bed friction. At time $t = t_0$ a pulse of tracer is introduced into the flow from a vertical line source. Tracer is carried downstream by the current and spreads out as a result of turbulent diffusion. Velocity shear causes tracer near the bed to be carried downstream more

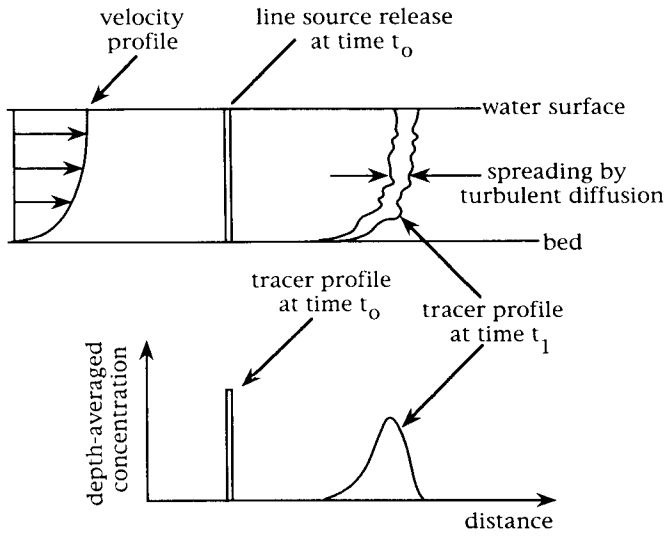


**Figure 1.10.** Sketch showing the combined effects of vertical velocity shear and turbulent diffusion on longitudinal spreading. A sudden release of tracer is made from a vertical line source at time $t_0$. At a later time $t_1$ the tracer cloud has been advected downstream, distorted by vertical velocity shear and smeared by turbulent diffusion

slowly than tracer near the surface and the initial straight line of tracer becomes distorted. Turbulent diffusion acts to smear and thicken the line of tracer. If we calculate the depth-averaged tracer concentration at $t = t_1$ we see that there has been considerable longitudinal spreading. Part of this spreading can be attributed to turbulent diffusion but most is attributable to vertical velocity shear. The terms in (1.83c) quantify the longitudinal and transverse tracer fluxes which result from vertical shear in the longitudinal and transverse velocities.

### 1.8.4 Taylor's Analysis of Turbulent Shear Flow

At this stage the dispersion terms (1.83c) are unknown and consequently equation (1.82) is intractable. We encountered a similar closure problem in Section 1.7.1. It would be possible to solve equation (1.82) if a relationship could be found between the dispersive flux and some property of the depth-averaged concentration. Taylor's asymptotic analysis of shear dispersion in a pipe furnishes such a relationship.

If turbulent diffusion was negligibly small then velocity shear would distort the tracer profile as shown in Figure 1.10 and would continue to distort it indefinitely. If the difference in velocity over the depth was $\Delta u_x$ then after time



$t$ tracer would be spread along the channel over a distance of the order

$$L = t\Delta u_x \tag{1.84}$$

Equation (1.84) suggests that the length of the cloud increases linearly with time. The length of the cloud is linearly related to the standard deviation (i.e. the square root of the variance) and so equation (1.84) implies that the variance of the tracer cloud increases as the square of time. In practice the variance increases at this rate only for a short period immediately after the tracer is released. This is the behaviour suggested by Taylor's analysis at small times [see equation (1.40)]. As the cloud evolves the rate of spreading decreases until at asymptotically long times the variance increases linearly with time [see equation (1.41)]. The reason for this change of behaviour is the interaction between velocity shear and turbulent diffusion. Vertical velocity shear distorts the initially straight line of tracer. This distortion increases the vertical concentration gradient and hence promotes vertical diffusion. Vertical diffusion counteracts the effects of vertical velocity shear and is the reason why equation (1.84) is not valid.

Taylor, whose 1921 paper is the key to quantifying turbulent diffusion, also elucidated the interaction between velocity shear and diffusion in steady laminar and then stationary turbulent pipe flow (Taylor 1953, 1954). He addressed the effects of radial velocity shear and radial diffusion on longitudinal mixing in a pipe, but subsequently his analysis was extended to the effects of both vertical (Elder 1959) and transverse velocity shear (Fischer 1966) on longitudinal mixing in open channels.

We will examine the effects of vertical velocity shear in open channel flow when tracer originates from an instantaneous transverse line source so that transverse concentration gradients are negligibly small. The channel is assumed to be uniform and the $x$ axis is aligned with the flow so that $u_y = u_z = 0$. The contribution made to the longitudinal spreading of the tracer cloud by turbulent diffusion is small compared with the effects of vertical velocity shear and consequently the term

$$\frac{\partial}{\partial x}\left(e_x \frac{\partial c}{\partial x}\right) \tag{1.85}$$

can be neglected. Making these assumptions equation (1.64) becomes

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} = \frac{\partial}{\partial y}\left(e_y \frac{\partial c}{\partial y}\right) \tag{1.86}$$

The velocity and concentration are now written in terms of their depth averages and deviations following equation (1.78) to give

$$c(x, y, t) = s(x, t) + c'(x, y, t) \tag{1.87a}$$

$$u_x(x, y) = v_x(x) + u'_x(x, y) \tag{1.87b}$$

Substituting from equation (1.87) in (1.86) and simplifying gives

$$\frac{\partial}{\partial t}(s+c')+(v_x+u_x')\frac{\partial}{\partial x}(s+c')=\frac{\partial}{\partial y}\left(e_z\frac{\partial c'}{\partial y}\right) \tag{1.88}$$

Note that $s$ does not vary with $y$ and so does not appear in the last term. Next, equation (1.88) is transformed into a Lagrangian coordinate system which travels downstream at mean velocity.

$$\xi = x - v_x t \tag{1.89a}$$

$$\tau = t \tag{1.89b}$$

Making use of the chain rule

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi} \tag{1.90a}$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}-v_x\frac{\partial}{\partial \xi} \tag{1.90b}$$

Equation (1.88) then becomes

$$\frac{\partial s}{\partial \tau}+\frac{\partial c'}{\partial \tau}+u_x'\frac{\partial s}{\partial \xi}+u_x'\frac{\partial c'}{\partial \xi}=\frac{\partial}{\partial y}\left(e_y\frac{\partial c'}{\partial y}\right) \tag{1.91}$$

A general solution to equation (1.91) cannot be found because $u_x'$ is a function of $y$. Taylor argued, however, that after tracer has been in the flow for a sufficiently long time an equilibrium becomes established between vertical velocity shear and turbulent diffusion. Once this equilibrium is established, the vertical distribution of the concentration deviation becomes independent of both $\tau$ and $\xi$. Thus

$$c' = c'(y) \tag{1.92}$$

In addition, the mean concentration only changes slowly so that

$$\frac{\partial s}{\partial \tau}=0 \tag{1.93}$$

Fischer *et al* (1979) determined the orders of magnitude of each of the terms in equation (1.91) and showed that it is justified to make these assumptions, while Sayre (1968a) showed using numerical experiments that at asymptotically large times equation (1.92) is valid. Equation (1.91) can then be simplified to

$$u_x'\frac{\partial s}{\partial \xi}=\frac{\partial}{\partial y}\left(e_y\frac{\partial c'}{\partial y}\right) \tag{1.94}$$



At the bed and water surface the vertical diffusive flux is zero

$$\frac{\partial c'}{\partial y}=0 \qquad \text{at} \qquad y=0 \text{ and } h \tag{1.95}$$

Integrating equation (1.94) twice with respect to $y$, making use of the boundary condition in equation (1.95) and noting that $s$ is independent of $y$

$$c'(y)=\frac{\partial s}{\partial \xi}\int_{y'=0}^{y}\frac{\partial y'}{e_y}\int_{y''=0}^{y'}u_x'(y'')\,\mathrm{d}y''+c'(0) \tag{1.96}$$

The depth-averaged dispersive flux in the longitudinal direction is

$$\frac{1}{h}\int_{y=0}^{h}u_x'c'\,\mathrm{d}y=\frac{\partial s}{\partial \xi}\frac{1}{h}\int_{y=0}^{h}u_x'\,\mathrm{d}y\int_{y'=0}^{y}\frac{\mathrm{d}y'}{e_y}\int_{y''=0}^{y'}u_x'\,\mathrm{d}y'' \tag{1.97}$$

Equation (1.97) implies that the longitudinal dispersive flux is proportional to the longitudinal gradient of the depth-averaged concentration. Thus at asymptotically long times longitudinal dispersion can be described using Fick's law

$$J_\xi = -k_x\frac{\partial s}{\partial \xi} \tag{1.98}$$

where $k_x$ is the longitudinal dispersion coefficient, which from (1.97) is

$$k_x=\frac{1}{h}\int_{y=0}^{h}u_x'\,\mathrm{d}y\int_{y'=0}^{y}\frac{\mathrm{d}y'}{e_y}\int_{y''=0}^{y'}u_x'\,\mathrm{d}y'' \tag{1.99}$$

By applying a mass balance to a water parcel which occupies the entire depth it is straightforward to show that

$$\frac{\partial s}{\partial \tau}=k_x\frac{\partial^2 s}{\partial \xi^2} \tag{1.100}$$

where $k_x$ is assumed constant. Reverting to Cartesian coordinates

$$\frac{\partial s}{\partial t}+v_x\frac{\partial s}{\partial x}=k_x\frac{\partial^2 s}{\partial x^2} \tag{1.101}$$

It is important to appreciate that equation (1.101) is an asymptotic result which only applies once an equilibrium becomes established between velocity shear and diffusion. In stationary homogeneous turbulence such an equilibrium is likely once an individual tracer particle has sampled the entire flow field. The time-scale for a tracer particle to sample the entire flow field in this analysis is the time-scale for vertical mixing. On dimensional grounds

this is

$$T_y = \frac{h^2}{e_y} \tag{1.102}$$

where $h$ = depth and $e_y$ = vertical turbulent diffusion coefficient.

If turbulence is non-stationary or non-homogeneous then we have difficulties applying Taylor's analysis. In this situation there may be insufficient time for an equilibrium to become established between velocity shear and turbulent diffusion before the tracer cloud is carried into a different part of the channel with a different turbulent regime.

### 1.8.5 Depth Averaging: Part Two

In Section 1.8.2 we found that equation (1.82) contained unknown dispersive fluxes. Taylor's analysis, as outlined in Section 1.8.4 shows that at asymptotically large times the longitudinal dispersive flux is proportional to the longitudinal gradient in depth-averaged concentration. Thus

$$-\overline{u_x' c'} = k_x \frac{\partial s}{\partial x} \tag{1.103}$$

where $k_x$ = the longitudinal dispersion coefficient which accounts for the effects on the depth-averaged tracer concentration of depth variations in the longitudinal velocity. By analogy with equation (1.103) the transverse dispersive flux is

$$-\overline{u_z' c'} = k_z \frac{\partial s}{\partial z} \tag{1.104}$$

where $k_z$ = the transverse dispersion coefficient which accounts for the effects on the depth-averaged tracer concentration of depth variations in the transverse velocity. Incorporating equations (1.103) and (1.104) into (1.82) yields

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}(h v_x s) + \frac{\partial}{\partial z}(h v_z s) = \frac{\partial}{\partial x}\left[h(e_x + k_x)\frac{\partial s}{\partial x}\right] + \frac{\partial}{\partial z}\left[h(e_z + k_z)\frac{\partial s}{\partial z}\right] \tag{1.105}$$

Invariably in river channels $k_x \gg e_x$ and $k_z \gg e_z$. Using the continuity equation (1.63), equation (1.105) can be rewritten in advection form as

$$\frac{\partial s}{\partial t} + v_x \frac{\partial s}{\partial x} + v_z \frac{\partial s}{\partial z} = \frac{1}{h}\frac{\partial}{\partial x}\left(h k_x \frac{\partial s}{\partial x}\right) + \frac{1}{h}\frac{\partial}{\partial z}\left(h k_z \frac{\partial s}{\partial z}\right) \tag{1.106}$$

Equation (1.106) is used extensively to solve river mixing problems in the mid-field and its practical application is described in Chapter 3.



### 1.8.6 Cross-section Averaging

A long way downstream from its point of injection tracer becomes spread across the whole channel. Beyond this point attention can be focused on the rate at which the cross-sectional averaged concentration is advected downstream and dispersed longitudinally. Longitudinal dispersion is only of interest when the input of tracer is non-steady (e.g. below a slug injection) because below a steady source the fully mixed concentration of a conservative tracer is constant. An equation governing the cross-sectional averaged concentration can be derived by integrating equation (1.106) across the channel which leads to

$$A\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(A V_x S) = \frac{\partial}{\partial x}\left(-A\,\overline{v_x' s'} + A k_x \frac{\partial S}{\partial x}\right) \tag{1.107}$$

where $S$ = cross-sectional averaged concentration, $A$ = cross-sectional area of the channel and $V_x$ = cross-sectional averaged velocity. The approach is similar to that used in Sections 1.8.2 and 1.8.5 and is left as an exercise for the reader.

$$S(x) = \frac{1}{b}\int_{z=0}^{b} s(x, z)\,\mathrm{d}z \tag{1.108a}$$

$$V_x(x) = \frac{1}{b}\int_{z=0}^{b} v_x(x, z)\,\mathrm{d}z \tag{1.108b}$$

$$s(x, z) = S(x) + s'(x, z) \tag{1.108c}$$

and

$$v_x(x, z) = V_x(x) + v_x'(x, z) \tag{1.108d}$$

where $b$ = channel width. It is assumed that $k_x$ is constant across the channel. The overbar denotes a width-average. The term

$$-\overline{v_x' s'} \tag{1.109}$$

is the longitudinal dispersive tracer flux resulting from variations in velocity across the channel. Taylor's analysis of turbulent shear flow (Section 1.8.4) suggests that at asymptotically large times this flux is proportional to the longitudinal gradient in cross-sectional averaged concentration. Thus

$$-\overline{v_x' s'} = K_x \frac{\partial S}{\partial x} \tag{1.110}$$

where $K_x$ is a constant. Making this substitution in equation (1.107) and noting

that invariably $K_x \gg k_x$ yields

$$\frac{\partial S}{\partial t} + V_x \frac{\partial S}{\partial x} = \frac{1}{A} \frac{\partial}{\partial x} \left( A K_x \frac{\partial S}{\partial x} \right) \tag{1.111}$$

where $K_x$ = the longitudinal dispersion coefficient which accounts for the effects on the cross-sectional averaged tracer concentration of variations of velocity across the channel cross-section. The practical application of equation (1.111) is discussed in Chapter 4.

## 1.9 CHANNEL CURVATURE

### 1.9.1 Cartesian Models

Where the channel is straight and uniform it is sensible to write the depth-averaged mixing equation in rectangular Cartesian coordinates (i.e. $x$–$z$ coordinates). This is clearly the best option in man-made channels such as canals and laboratory flumes and is sometimes satisfactory in the mid-field of natural rivers. In such situations the depth-averaged tracer concentration can be predicted using

$$\frac{\partial s}{\partial t} + v_x \frac{\partial s}{\partial x} + v_z \frac{\partial s}{\partial z} = \frac{1}{h} \frac{\partial}{\partial x} \left( h k_x \frac{\partial s}{\partial x} \right) + \frac{1}{h} \frac{\partial}{\partial z} \left( h k_z \frac{\partial s}{\partial z} \right) \tag{1.106}$$

In many rivers, however, the channel meanders, the banks are not straight and the cross-section is highly non-uniform. There are several difficulties using equation (1.106) in such channels. Firstly, there are good reasons for believing that in rivers the longitudinal and transverse dispersion coefficients are non-homogeneous ($k_x \neq k_y$). In a meandering channel it is not possible to arrange for the cartesian $x$ axis always to be aligned with the main flow direction nor the cartesian $z$ axis transversely across the main flow. Consequently, $k_x$ and $k_z$ are not always the longitudinal and transverse dispersion coefficients. Secondly, in a meandering channel the $x$ and $z$ axes are no longer always aligned with the principal axes of the diffusion tensor. If the longitudinal and transverse diffusivities are different then (as discussed in Section 1.7.7) it is necessary to include cross-derivative dispersion terms in the mixing equation (see equation 1.67). Both of these problems disappear if $k_x = k_y$. This assumption can safely be made if the tracer source is steady because in this situation longitudinal dispersion is unimportant. Thirdly, at the banks the usual boundary condition is that the tracer flux normal to the bank is zero. In a meandering channel the normal flux is a complicated function involving derivatives in both $x$ and $z$ directions which makes solving equation (1.106) difficult.

One method to overcome these problems is to align the $x$ axis with the main flow path. The $x$ axis is then no longer straight but follows the meander



pattern of the channel. The $z$ axis is taken everywhere orthogonal to the $x$ axis. Distance $x$ is measured in the local downstream direction and distance $z$ is measured in the local transverse direction. This leads to a curvilinear coordinate system (Figure 1.11).

As a first approximation the fact that the downstream and transverse directions vary from place to place along the channel can be ignored and equation (1.106) can be retained. This approach works well in gently meandering rivers (Rutherford and Williams 1992). Equation (1.106) is, however, no longer exact because curvature distorts the coordinate system.

### 1.9.2 Curvilinear Models

A more rigorous approach is to transform the equation governing tracer concentration from the rectangular Cartesian coordinate system to a curvilinear system. Consider the problem of predicting the depth-averaged tracer concentration in a meandering channel. Figure 1.11 shows such a channel in plan view together with curvilinear coordinate axes denoted $\alpha$ (longitudinal) and $\beta$ (transverse). The $\alpha$ and $\beta$ lines are curved, at right angles to each other (orthogonal) and the $\alpha$ lines are parallel with the local depth-averaged velocity vectors. These two assumptions mean that several terms disappear from the mixing equation during the coordinate transformation leading to a simpler model. Because of curvature and/or variations in channel width, longitudinal distances for a given increment in $\alpha$ vary depending on the value of $\beta$. Thus in Figure 1.11 the distance $L_{AB}$ is not in general equal to the distance $L_{CD}$. Similarly transverse distances for a given increment in $\beta$ vary depending on the value of $\alpha$. Thus the distance $L_{AC}$ is not in general equal to the distance $L_{BD}$. Metric coefficients are defined which relate distances measured along the

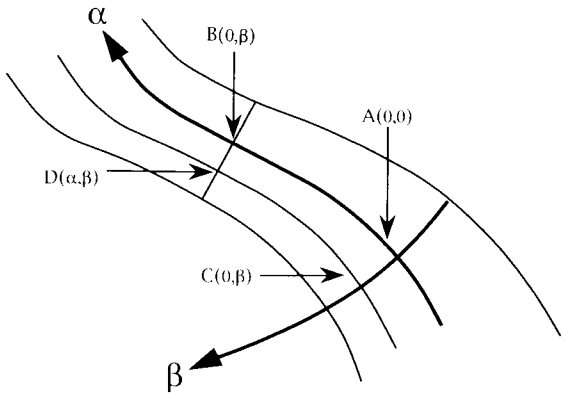


**Figure 1.11.** Sketch showing an orthogonal curvilinear coordinate system in a meandering river channel

coordinate axes to distances measured along other $\alpha$ and $\beta$ lines. The metric coefficients are functions of $\alpha$ and $\beta$. In Figure 1.11 the origin of the curvilinear coordinate system is located at A. The coordinates of points A, B, C and D are respectively $(0, 0)$, $(\alpha, 0)$, $(0, \beta)$, $(\alpha, \beta)$. The average value of the longitudinal metric coefficient $m_\alpha$ along the line C-D is

$$\overline{m_\alpha} = \frac{L_{CD}}{L_{AB}} \tag{1.112}$$

or, more generally

$$L_{CD} = \int_{\alpha'=0}^{\alpha} m_\alpha(\alpha', \beta)\, d\alpha' \tag{1.113}$$

Similarly the average value of the transverse metric coefficient $m_\beta$ along the line B-D is

$$\overline{m_\beta} = \frac{L_{BD}}{L_{AC}} \tag{1.114}$$

or, more generally

$$L_{BD} = \int_{\beta'=0}^{\beta} m_\beta(\alpha, \beta')\, d\beta' \tag{1.115}$$

The rectangular Cartesian coordinate system is a special case in which $m_\alpha = m_\beta = 1$. In curved channels the metric coefficients are typically of the order of 0.8–1.2 (Holly 1985).

The derivation of the advection/dispersion equation in curvilinear coordinates is given in some detail by Chang (1971) and Yotsukura (1977) and so is not repeated here. The depth-averaged equations of continuity and tracer mass balance for unsteady flow are simply stated

$$m_\alpha m_\beta \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(m_\beta h v_\alpha) + \frac{\partial}{\partial z}(m_\alpha h v_\beta) = 0 \tag{1.116}$$

$$\frac{\partial s}{\partial t} + \frac{v_\alpha}{m_\alpha}\frac{\partial s}{\partial \alpha} + \frac{v_\beta}{m_\beta}\frac{\partial s}{\partial \beta} = \frac{1}{h m_\alpha m_\beta}\left[\frac{\partial}{\partial \alpha}\left(\frac{m_\beta}{m_\alpha} h k_\alpha \frac{\partial s}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{m_\alpha}{m_\beta} h k_\beta \frac{\partial s}{\partial \beta}\right)\right] \tag{1.117}$$

where $\alpha$ and $\beta$ = longitudinal and transverse curvilinear distances; $h$ = local water depth; $v_\alpha$ and $v_\beta$ = depth-averaged longitudinal and transverse velocities; $s$ = depth-averaged concentration; $k_\alpha$ and $k_\beta$ = longitudinal and transverse dispersion coefficients (which quantify the effects of turbulent diffusion and depth variations in longitudinal and transverse velocity); and $m_\alpha$ and $m_\beta$ = metric coefficients.



### 1.9.3 Streamtube Model

It is possible to simplify equation (1.117) in the situation where the river flow is steady. Firstly, the time varying term disappears. We can also define a new variable, the cumulative discharge

$$q(\alpha, \beta) = \int_{\beta'=-\infty}^{\beta} m_\beta h v_\alpha\, d\beta' \tag{1.118}$$

where $q(\alpha, \beta)$ = cumulative discharge; $h(\alpha, \beta)$ = local depth; $v_\alpha(\alpha, \beta)$ = local depth-averaged longitudinal velocity; and $m_\beta(\alpha, \beta)$ = transverse metric coefficient. If the left and right banks are located at $\beta_l$ and $\beta_r$ then $q(\alpha, \beta_l) = 0$ and $q(\alpha, \beta_r) = Q$ (for all values of $\alpha$) where $Q$ = total river flow.

Now equation (1.117) can be integrated with respect to $\beta$ starting from the left bank. The approach is similar to that outlined in Section 1.8.2. Use is made of Leibnitz's rule (equation 1.69), the chain rule (equation 1.66) and the boundary condition $v_\alpha = v_\beta = 0$ at the left and right banks. The interested reader is referred to Yotsukura (1977) for full details of the derivation. For an unsteady tracer source in steady river flow the conservation of mass equation becomes

$$\frac{\partial s}{\partial t} + \frac{v_\alpha}{m_\alpha}\frac{\partial s}{\partial \alpha} = \frac{1}{h m_\alpha m_\beta}\frac{\partial}{\partial \alpha}\left(\frac{m_\beta}{m_\alpha} h k_\alpha \frac{\partial s}{\partial \alpha}\right) + \frac{v_\alpha}{m_\alpha}\frac{\partial}{\partial q}\left(m_\alpha h^2 v_\alpha k_\beta \frac{\partial s}{\partial q}\right) \tag{1.119}$$

If the tracer source is steady the unsteady term (term 1) vanishes and the longitudinal dispersion term (term 3) becomes negligibly small (Yotsukura and Sayre 1976) giving

$$\frac{\partial s}{\partial \alpha} = \frac{\partial}{\partial q}\left(m_\alpha h^2 v_\alpha k_\beta \frac{\partial s}{\partial q}\right) \tag{1.120}$$

Equations (1.119) and (1.120) contain the product

$$D(\alpha, q) = m_\alpha h^2 v_\alpha k_\beta \tag{1.121}$$

$D$ is called the factor of diffusion and has the units $m^5 s^{-2}$. Thus $D/Q^2$ has the units $m^{-1}$ and is an inverse length scale. The factor of diffusion varies across the channel because $h$ and $v_\alpha$ are both low near the banks. The great advantage of equations (1.119) and (1.120) is that they do not contain the transverse velocity $v_\beta$ explicitly although they still take account of cross-channel water movements. The practical application of the stream tube model is described in Chapter 3.

## NOTATION

| | |
|---|---|
| $a, b$ | location of the water surface and river bed |
| $A$ | cross-sectional area |
| $b$ | channel width |
| $c$ | tracer concentration |
| $\bar{c}$ | mean concentration |
| $c'$ | concentration deviation from the ensemble or depth average |
| $\langle c \rangle$ | ensemble mean tracer concentration |
| $c_i$ | realisation $i$ of the tracer concentration |
| $c_t$ | average concentration in a parcel at time $t$ |
| D | factor of diffusion |
| $e_m$ | molecular diffusion coefficient |
| $e_t$ | turbulent diffusion coefficient (or eddy diffusivity) |
| $e_x, e_y, e_z$ | eddy diffusivities in the $x$, $y$, $z$ directions |
| $e_{ij}$ | elements of the eddy diffusivity tensor |
| $h$ | local depth |
| $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ | unit normal vectors in the $x$, $y$, $z$ directions |
| $I_x$ | advective flux in the $x$ direction |
| $\boldsymbol{J}$ | tracer flux vector |
| $J_x, J_y, J_z$ | diffusive fluxes in the $x$, $y$, $z$ directions |
| $J_\xi$ | diffusive flux in a moving coordinate system |
| $k_\alpha, k_\beta$ | longitudinal and transverse dispersion coefficients in depth-averaged curvilinear mixing equations |
| $k_x, k_z$ | longitudinal and transverse dispersion coefficients in depth-averaged mixing equations |
| $K_x$ | longitudinal dispersion coefficient in cross-sectional averaged mixing equations |
| $L$ | a characteristic length scale |
| $L_{AB}$ | distance along a streamline between points A and B |
| $L_g$ | mean free path of a gas molecule |
| $L_L$ | integral length scale |
| $L_m$ | Prandtl's turbulent mixing length |
| $L_x$ | integral length scale in the $x$ direction |
| $m$ | mass inflow rate of tracer |
| $m_\alpha, m_\beta$ | metric coefficients for a curvilinear coordinate system |
| $M$ | mass of tracer injected |
| $M_t$ | tracer mass in a fluid parcel at time $t$ |
| $n_a, n_b$ | vectors normal to the water surface and bed |
| $N$ | number of realisations |
| $Re$ | Reynolds number |
| $R_L$ | Lagrangian autocorrelation function |



| | |
|---|---|
| $Q$ | cumulative flow |
| $Q$ | total river flow |
| $s$ | depth-averaged concentration |
| $s'$ | deviation from the depth-averaged concentration |
| $S$ | cross-sectional averaged concentration |
| $Sc_t$ | turbulent Schmidt number $= e_t/\nu_t$ |
| $t$ | time |
| $\Delta t$ | time increment |
| $t_0$ | fixed time |
| $T$ | time period for averaging |
| $T_L$ | integral Lagrangian time scale in the $x$ direction |
| $T_L$ | time scale for vertical mixing |
| $u$ | a characteristic velocity |
| $u_m$ | typical molecular velocity of a gas |
| $u_t$ | turbulent velocity |
| $u_x, u_y, u_z$ | velocities in the $x$, $y$, $z$ directions |
| $\langle u_x \rangle, \langle u_y \rangle, \langle u_z \rangle$ | ensemble mean velocities in the $x$, $y$, $z$ directions |
| $\Delta u$ | velocity difference |
| $u_x', u_y', u_z'$ | velocity deviation from the depth or ensemble average |
| $u_x, u_z$ | depth-averaged velocities in the $x$ and $z$ directions |
| $u_\alpha, u_\beta$ | depth-averaged velocities in the $\alpha$, $\beta$ directions |
| $V$ | volume |
| $V_x$ | cross-sectional averaged velocity |
| $x, y, z$ | longitudinal, vertical and transverse distances |
| $x_c, y_c, z_c$ | centroid of an individual tracer cloud |
| $\langle x \rangle, \langle y \rangle, \langle z \rangle$ | centroid of an ensemble of tracer clouds |
| $\Delta x, \Delta y, \Delta z$ | incremental distances in the $x$, $y$, $z$ directions |
| $x_0, y_0, z_0$ | fixed location |
| $\alpha, \beta, \gamma$ | rotated Cartesian coordinate axes |
| $\alpha, \beta$ | curvilinear coordinate directions |
| $\Lambda_x, \ldots$ | eddy length scales |
| $\eta$ | location of the water surface or the bed |
| $\mu$ | dynamic viscosity |
| $\nu$ | kinematic viscosity |
| $\nu_t$ | eddy viscosity |
| $\nu_{tg}$ | eddy viscosity of a gas |
| $\xi$ | longitudinal distance in a moving Lagrangian coordinate system |
| $\rho$ | density |
| $\sigma_x^2, \sigma_y^2, \sigma_z^2$ | tracer cloud variances in the $x$, $y$, $z$ direction about the centre of mass of the individual tracer cloud |
| $\Sigma_x^2, \Sigma_y^2, \Sigma_z^2$ | variances about the ensemble mean centre of mass |

| | |
|---|---|
| $\tau$ | viscous shear stress |
| $\tau_t$ | turbulent shear stress (Reynolds stress) |
| $\tau$ | time |

# Chapter 2

# Vertical Mixing

## 2.1 INTRODUCTION

In rivers, the turbulence generated by bed friction is strong and the depth is generally small so that a tracer becomes vertically well-mixed within a short distance of the source. This point is illustarted in Figure 1.9 and Table 1.2. This chapter concentrates on the near-field and quantifies the rate of vertical mixing. It begins with a discussion of Prandtl's mixing length hypothesis (which was introduced briefly in Chapter 1) and shows how it provides expressions for the vertical variation of the ensemble mean velocity and the vertical diffusivity. An approximate model is then introduced for vertical mixing downstream from a steady transverse line source which assumes that the velocity and diffusivity are constant. This model admits analytical solutions and can be used to derive fairly simple solutions to a large number of vertical mixing problems. One of the advantages of the constant-coefficient model is that solutions to instantaneous point source problems can be superposed to address problems of vertical mixing downstream from sources with complex geometry and/or time-varying discharge. These solutions are useful for understanding the general behaviour of the tracer plume as well as for making approximate calculations of tracer concentration. A more realistic, but more complex, model is then introduced in which the velocity varies logarithmically and the diffusivity varies parabolically over the depth. There are slight differences between the variable- and constant-coefficient models which are discussed. For most of this chapter tracers are assumed to be neutrally buoyant, but there is a brief discussion of the effects of buoyancy on vertical mixing.

## 2.2 SUMMARY

This summary is placed at the beginning of the chapter to help the reader appreciate where the ensuing discussions are leading and to act as a checklist subsequently.