PRIMO COMPITO INTERMEDIO

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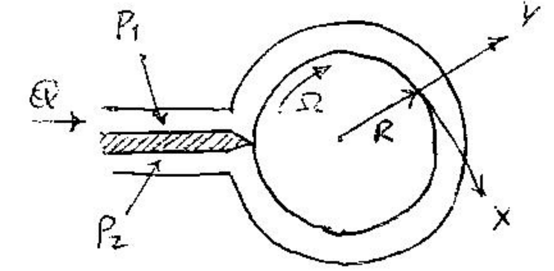
$\boxed{\text{ESE } 1}$



Ipotesi semplificative:

1. Moto stazionario: $\frac{\partial}{\partial t} = 0$
2. Moto bidimensionale: $\frac{\partial}{\partial z} = 0$
3. Moto unidirezionale: $v_x(y) \neq 0$

   $v_y = v_z = 0$
4. Fluido newtoniano, incomprimibile

Continuità: $\frac{\partial v_x}{\partial x} + \cancelto{0}{\frac{\partial v_y}{\partial y}} + \cancelto{0}{\frac{\partial v_z}{\partial z}} = 0 \Rightarrow \frac{\partial v_x}{\partial x} = 0$ Flusso completamente sviluppato

NS_x : $0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2}$ con $\frac{\partial P}{\partial x} = \frac{\Delta P}{L}\Big|_{L = 2\pi R}$

NS_y : $0 = -\frac{\partial P}{\partial y}$

$\frac{\partial v_x}{\partial y} = \frac{1}{\mu}\left(\frac{\Delta P}{L}\right) y + C_1$, $v_x(y) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right) y^2 + C_1 y + C_2$

C.C. #1 : vel. nulla sulla parete interna del cilindro esterno

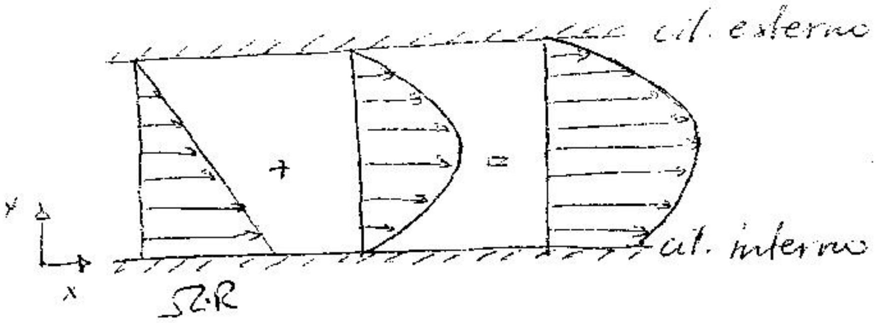
$$v_x(y = h) = 0 \Rightarrow 0 = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right) h^2 + C_1 h + C_2 \quad (1)$$

C.C. #2 : vel. pari ad $\Omega \cdot R$ sulla parete interna del cilindro interno

$$v_x(y = 0) = \Omega \cdot R \Rightarrow \Omega R = C_2 \quad (2)$$

Sostituendo la (2) nella (1): $C_1 = -\frac{1}{2\mu}\left(\frac{\Delta P}{L}\right) h - \frac{\Omega R}{h}$

Pertanto: $\boxed{v_x(y) = \frac{1}{2\mu}\left(-\frac{\Delta P}{L}\right)(hy - y^2) + \Omega R\left(1 - y/h\right)}$ PROFILO DI VELOCITA'



ANDAMENTO QUALITATIVO DEL PROFILO DI VELOCITA'

La vel. risulta massima per $y^* = \frac{1}{2}\left[h - \frac{2\mu\Omega R}{h}\cdot\frac{1}{(-\Delta P/L)}\right]$

Infatti: $v_x(y) = v_{x,max}$ se $\frac{\partial v_x}{\partial y} = 0 \Rightarrow \frac{1}{2\mu}\left(-\frac{\Delta P}{L}\right)(h-2y) - \frac{\Omega R}{h} = 0$

$$h - 2y = \frac{\Omega R}{h}\cdot\frac{2\mu}{(-\Delta P/L)} \Rightarrow y = \frac{1}{2}\left[h - \frac{2\mu\Omega R}{h}\cdot\frac{1}{(-\Delta P/L)}\right]$$

Per avere $v_{x,max} = v_x(y=0)$ deve quindi risultare:

$$y = \frac{h}{2} \Rightarrow h - \frac{2\mu\Omega R}{h}\cdot\frac{1}{(-\Delta P/L)} = \frac{h}{2} \Rightarrow \frac{h}{2}\left(-\frac{\Delta P}{L}\right) = \frac{2\mu\Omega R}{h}$$

$$\Omega = \frac{h}{2}\left(-\frac{\Delta P}{L}\right)\cdot\frac{h}{2\mu R} = \left(-\frac{\Delta P}{L}\right)\frac{h^2}{4\mu R} \Rightarrow \boxed{\Omega = \left(-\frac{\Delta P}{L}\right)\cdot\frac{h^2}{4\mu R}} \quad (3)$$

Se vale la (3): $\underline{\underline{v_{x,max}}} = \frac{1}{2\mu}\left(-\frac{\Delta P}{L}\right)\underbrace{\left(\frac{h^2}{2} - \frac{h^2}{4}\right)}_{h^2/4} + \left(-\frac{\Delta P}{L}\right)\frac{h^2}{4\mu \cancel{R}}\cdot \cancel{R}\underbrace{\left(1 - \frac{1}{2}\right)}_{1/2}$

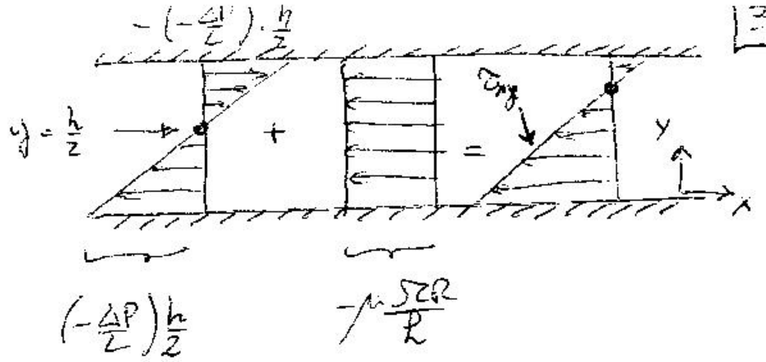
$$= \frac{1}{8\mu}\left(-\frac{\Delta P}{L}\right)h^2 + \frac{1}{8\mu}\left(-\frac{\Delta P}{L}\right)h^2 = \underline{\underline{\frac{1}{4\mu}\left(-\frac{\Delta P}{L}\right)h^2}}$$

Andamento del taglio:

$$\tau_{xy} = \mu\frac{\partial v_x}{\partial y} = \mu\left[\frac{1}{2\mu}\left(-\frac{\Delta P}{L}\right)(h-2y) - \frac{\Omega R}{h}\right] = \left(-\frac{\Delta P}{L}\right)\frac{h-2y}{2} - \frac{\mu\Omega R}{h}$$

$$\tau_w = \tau_{xy}\big|_{y=0} = \left(-\frac{\Delta P}{L}\right)\cdot\frac{h}{2} - \frac{\mu\Omega R}{h}$$

$$\boxed{\tau_{xy} = \left(-\frac{\Delta P}{L}\right)\left(\frac{h}{2} - y\right) - \mu\frac{\Omega R}{h}}$$



Calcolo della forza F : $F = \tau_{xy}\big|_{y=0} \cdot LW = \tau_w \cdot LW =$

$$= \left[\left(-\frac{\Delta P}{L}\right)\frac{h}{2} - \mu\frac{\Omega R}{h}\right]LW$$

$$Pot = F \cdot \Omega R = \left[\left(-\frac{\Delta P}{L}\right)\cdot\frac{h}{2} - \mu\frac{\Omega R}{h}\right]\Omega R L W$$

$$= \left[\underbrace{\left(-\frac{\Delta P}{L}\right)\frac{h}{2} - \cancel{\frac{\mu R}{h}}\cdot\left(-\frac{\Delta P}{L}\right)\frac{h^2}{\cancel{4\mu R}}}\right]\left(-\frac{\Delta P}{L}\right)\cdot\frac{h^2}{4\mu \cancel{R}}\cdot \cancel{R} L W$$

$$+\left(-\frac{\Delta P}{L}\right)\frac{h}{2} - \left(-\frac{\Delta P}{L}\right)\frac{h}{4} = \left(-\frac{\Delta P}{L}\right)\frac{h}{4}$$

$$= \left(-\frac{\Delta P}{L}\right)^2\frac{h^3}{16\mu}\cdot LW \qquad \left[\frac{Pa^{\cancel{2}}}{\cancel{m^2}}\cdot\frac{m^3}{\cancel{Pa}\cdot s}\cdot\cancel{m^2} = \frac{N}{m^2}\cdot\frac{m^3}{s} = \frac{N\cdot m}{s} = \frac{J}{s} = W\right]$$

$$\boxed{Pot = \left[\left(-\frac{\Delta P}{L}\right)^2\cdot\frac{h^3}{16\mu}\right]\cdot LW}$$

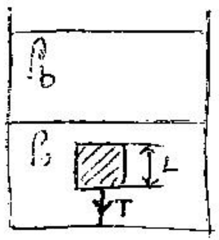
POTENZA SPESA PER MANTENERE IN MOTO IL CILINDRO INTERNO

<u>NOTA</u>: Se il riferimento cartesiano ha l'origine sul cilindro esterno:

$$\boxed{v_x(y) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)\left(y^2 - hy\right) + \frac{\Omega R}{h}y}$$

$$\boxed{\tau_{xy} = \left(\frac{\Delta P}{L}\right)\left(\frac{h}{2} - y\right) + \mu\frac{\Omega R}{h}} \qquad \boxed{Pot = \left[-\left(\frac{\Delta P}{L}\right)^2\frac{h^3}{16\mu}\right]LW}$$

EXE 2



Nella configurazione di equilibrio 1, il bilancio di forze risulta:

$$\vec{F}_p + \vec{F}_{gall} + \vec{T} = \vec{0} \qquad (1)$$

L'eq. vettoriale (1) ha un'unica comp. scalare in direzione verticale: $F_{gall} = F_p + T$ (se $\rho_o > \rho$).

Da tale eq. si ricava: $\rho_o g V = \rho g V + T \Rightarrow T = (\rho_o - \rho) g V = (\rho_o - \rho) g \ell^3$

$$\boxed{T = (\rho_o - \rho) g \ell^3}$$

$$\boxed{\rho = \rho_o - \frac{T}{g \ell^3}}$$

Nella configurazione di equilibrio 2, il bilancio di forze in forma scalare risulta:

$$F_p = F_{gall} \Rightarrow \rho g V = \rho_o g V_{olio} + \rho_b g V_{benzina}$$

con $V_{olio} = \ell^2 \cdot x$ e $V_{benzina} = \ell^2 \cdot (\ell - x)$. Quindi:

$$\rho \cancel{g} \ell^{\cancel{3}} = \rho_o \cancel{g} \cancel{\ell^2} \cdot x + \rho_b \cancel{g} \cancel{\ell^2} (\ell - x) \Rightarrow \rho \ell = \rho_o \cdot x + \rho_b (\ell - x)$$

$$\rho \ell = \rho_o \cdot x + \rho_b \cdot \ell - \rho_b \cdot x = \rho_b \cdot \ell + x(\rho_o - \rho_b)$$

$$\boxed{x = \frac{(\rho - \rho_b)\ell}{(\rho_o - \rho_b)} = \frac{\rho - \rho_b}{\rho_o - \rho_b} \cdot \ell} \qquad \boxed{\ell - x = \frac{\rho_o - \rho}{\rho_o - \rho_b} \ell}$$

Una via alternativa per arrivare alla medesima conclusione è la seguente:

- poiché $\rho < \rho_0$ allora $F_{peso} < F_{gall} \equiv \rho g V < \rho_0 g V$ (1)
  Nella configurazione di equilibrio 2, l'eq. (1) diventa:
  $\Delta F' = F_{gall} - F_{peso} = (\rho_0 - \rho) g\, l^2 \cdot x$ per la frazione di boa immersa nell'olio

- poiché $\rho_b < \rho$ allora $F_{gall} < F_{peso} \equiv \rho_b g V < \rho g V$ (2)
  Nella configurazione di equilibrio 2, l'eq (2) diventa:
  $\Delta F'' = F_{peso} - F_{gall} = (\rho - \rho_b) g\, l^2 \cdot (l - x)$ per la frazione di boa immersa nella benzina

In condizioni di equilibrio statico, deve essere:

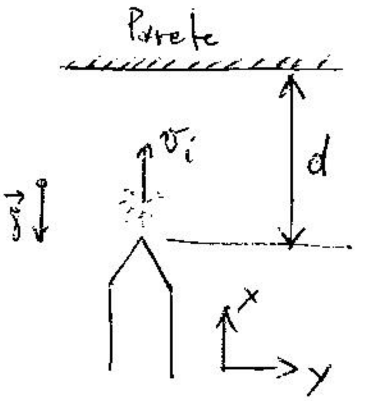
$$\Delta F' = \Delta F'' \Rightarrow (\rho_0 - \rho)\cancel{g l^2} \cdot x = (\rho - \rho_b)\cancel{g l^2}(l - x)$$

$$\boxed{x = \left(\frac{\rho - \rho_b}{\rho_0 - \rho_b}\right) \cdot l}$$ FRAZIONE DI BOA IMMERSA IN OLIO

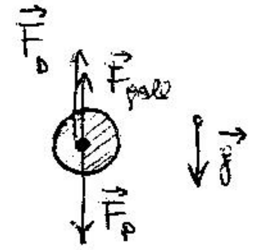
Se x rappresenta invece la frazione di boa immersa nella benzina, allora si ha:

$$\boxed{x = l - \frac{\rho - \rho_b}{\rho_0 - \rho_b} l = \left(\frac{\rho_0 - \rho}{\rho_0 - \rho_b}\right) l}$$ FRAZIONE DI BOA IMMERSA IN BENZINA

EXE 3



- ρ_p, D_p, $v_p(t=0) = v_i$
- $C_D = \frac{24}{Re_p}$
- $\rho_p > \rho \Rightarrow F_{peso} > F_{gall}$!
- Bilancio di forze: $\vec F_I = \vec F_D + \vec F_p + \vec F_{gall}$



$$m_p \frac{d\vec v_p}{dt} = \frac{1}{2}\rho C_D \frac{\pi D_p^2}{4}(\vec u - \vec v_p)|\vec u - \vec v_p| + \rho_p \vec g V - \rho \vec g V$$

In direzione verticale, il bilancio di forze in forma scalare:

$$m_p \frac{dv_p}{dt} = \frac{1}{2}\rho \frac{24}{\frac{\rho \cancel{|u-v_p|} D_p}{\mu}} \cdot \frac{\pi D_p^2}{4}(u^0 - v_p)\cancel{|u-v_p|} + (\rho - \rho_p) g V$$

$$= -3\pi\mu D_p v_p + (\rho - \rho_p) g V$$

$$(1) \quad \boxed{\frac{dv_p}{dt} = -\frac{v_p}{\tau_p} + \hat\rho g}$$ con $\tau_p = \frac{\rho_p D_p^2}{18\mu}$ e $\hat\rho = \frac{\rho - \rho_p}{\rho_p}$ (<0)

Risoluzione dell'eq. (1):

A) Soluz. particolare: $\frac{dv_p}{dt} = 0 \Rightarrow v_p^* = \hat\rho g \tau_p$

B) Soluz. omogenea associata: $\frac{dv_p}{dt} = -\frac{v_p}{\tau_p} \Rightarrow \frac{dv_p}{v_p} = -\frac{dt}{\tau_p}$

$$\ln v_p = -\frac{t}{\tau_p} + c_1 \Rightarrow v_p^{**} = e^{-t/\tau_p + c_1} = c_2 e^{-t/\tau_p} \quad (c_2 = e^{c_1})$$

C) Soluz. generale: $v_p(t) = v_p^* + v_p^{**} = c_2 e^{-t/\tau_p} + \hat\rho g \tau_p$

C.I. $v_p(t=0) = v_i \Rightarrow v_i = c_2 \underbrace{e^{-t/\tau_p}\big|_{t=0}}_{=1} + \hat\rho g \tau_p \Rightarrow \boxed{c_2 = v_i - \hat\rho g \tau_p}$

Pertanto:

$$\boxed{v_p(t) = v_i e^{-t/\tau_p} + \hat\rho g \tau_p (1 - e^{-t/\tau_p})}$$

$$X_p(t) = \int_0^t v_p(t)\,dt = v_i \int_0^t e^{-t/\tau_p}dt + \hat{\rho} g \tau_p \int_0^t (1 - e^{-t/\tau_p})dt$$

$$= v_i(-\tau_p)\cdot e^{-t/\tau_p}\Big|_0^t + \hat{\rho} g \tau_p \left[t - (-\tau_p) e^{-t/\tau_p}\Big|_0^t \right] =$$

$$= -v_i \tau_p (e^{-t/\tau_p} - 1) + \hat{\rho} g \tau_p \left[t + \tau_p (e^{-t/\tau_p} - 1) \right]$$

$$\boxed{X_p(t) = (v_i \tau_p - \hat{\rho} g \tau_p^2)(1 - e^{-t/\tau_p}) + \hat{\rho} g \tau_p \cdot t}$$

La massima distanza coperta dalle goccioline in direzione verticale corrisponde alla condizione $v_p(t)=0$:

$$v_p(t) = 0 \text{ se } \quad 0 = v_i e^{-t/\tau_p} + \hat{\rho} g \tau_p (1 - e^{-t/\tau_p})$$

$$0 = e^{-t/\tau_p}(v_i - \hat{\rho} g \tau_p) + \hat{\rho} g \tau_p$$

$$\boxed{t = \ln\left(1 - \frac{v_i}{\hat{\rho} g \tau_p}\right)^{\tau_p}} \Leftarrow e^{-t/\tau_p} = \frac{\hat{\rho} g \tau_p}{\hat{\rho} g \tau_p - v_i} \Rightarrow \boxed{1 - e^{-t/\tau_p} = -\frac{v_i}{\hat{\rho} g \tau_p - v_i}}$$

Quindi $X_{p,max} = (v_i \tau_p - \hat{\rho} g \tau_p^2)\cdot\left(-\frac{v_i}{\hat{\rho} g \tau_p - v_i}\right) + \hat{\rho} g \tau_p \ln\left(1 - \frac{v_i}{\hat{\rho} g \tau_p}\right)^{\tau_p}$

$$\underbrace{\tau_p \cancel{(v_i - \hat{\rho} g \tau_p)}\cdot\left(+\frac{v_i}{\cancel{v_i - \hat{\rho} g \tau_p}}\right)}$$

$$= v_i \tau_p + \hat{\rho} g \tau_p \ln\left(1 - \frac{v_i}{\hat{\rho} g \tau_p}\right)^{\tau_p}$$

$$\boxed{X_{p,max} = v_i \tau_p + \hat{\rho} g \tau_p \ln\left(1 - \frac{v_i}{\hat{\rho} g \tau_p}\right)^{\tau_p}}$$

Se, invece, $\tau \gg \tau_p$ allora $\tau/\tau_p \gg 1$ e $e^{-t/\tau_p} \simeq 0$. Quindi:

$$X_p(t) \simeq v_i \tau_p - \hat{\rho} g \tau_p^2 + \hat{\rho} g \tau_p \cdot t$$

Affinché sia $x_p(t) = d$, il tempo t richiesto è:

$$d = v_i \tau_p - \hat{\rho} g \tau_p^2 + \hat{\rho} g \tau_p \cdot t \Rightarrow t = \frac{d - v_i \tau_p + \hat{\rho} g \tau_p^2}{\hat{\rho} g \tau_p}$$

$$\boxed{t = \tau_p + \frac{d - v_i \tau_p}{\hat{\rho} g \tau_p}}$$