II) TURBULENT B.L. ON A FLAT PLATE

In the region where the B.L. is turbulent the following procedure can be followed. Starting from the NS_x in the case $\partial p / \partial x = 0$, one can integrate such equation along the vertical direction y, from the wall ($y=0$) to a distance far away from it ($y \to \infty$):

$$\underbrace{\int_0^{\infty} v_x \frac{\partial v_x}{\partial x} dy}_{A} + \underbrace{\int_0^{\infty} v_y \frac{\partial v_x}{\partial y} dy}_{B} = \underbrace{\nu \int_0^{\infty} \frac{\partial^2 v_x}{\partial y^2} dy}_{C} \quad [11]$$

Integral B can be recast as:

$$\int_0^{\infty} v_y \frac{\partial v_x}{\partial y} dy = \underbrace{v_x \cdot v_y \Big|_0^{\infty}} - \int_0^{\infty} v_x \frac{\partial v_y}{\partial y} dy$$

$$\underbrace{v_x(y \to \infty)}_{= v_\infty} \cdot v_y(y \to \infty) - \cancel{v_x(y=0) \cdot v_y(y=0)} \to 0$$

$$= v_\infty \cdot v_y(y \to \infty) - \int_0^{\infty} v_x \frac{\partial v_y}{\partial y} dy$$

Now, what is $v_y(y \to \infty) = ?$ We can find out using continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \Rightarrow \frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x} \Rightarrow d v_y = -\frac{\partial v_x}{\partial x} dy$$

$$\underbrace{\int_{v_y(y=0)}^{v_y(y\to\infty)} dv_y}_{v_y(y\to\infty) - \cancel{v_y(y=0)}^{=0}} = -\int_0^\infty \frac{\partial v_x}{\partial x} dy \Rightarrow \boxed{v_y(y\to\infty) = -\int_0^\infty \frac{\partial v_x}{\partial x} dy}$$

The condition $\frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x}$ also yields:

$$-\int_0^\infty v_x \frac{\partial v_y}{\partial y} dy = +\int_0^\infty v_x \frac{\partial v_x}{\partial x} dy$$

Integral (B) can thus be rewritten as:

$$\int_0^\infty v_y \frac{\partial v_x}{\partial y} dy = -v_\infty \int_0^\infty \frac{\partial v_x}{\partial x} dy + \int_0^\infty v_x \frac{\partial v_x}{\partial x} dy$$

Integral (C) can be recast as:

$$\mu \int_0^\infty \frac{\partial^2 v_x}{\partial y^2} dy = \mu \cdot \frac{\partial v_x}{\partial y}\bigg|_0^\infty = \mu \left(\cancel{\frac{\partial v_x}{\partial y}\bigg|_{y\to\infty}}^{=0 \ (v_x = v_\infty = \text{const} \ @ \ y\to\infty)} - \frac{\partial v_x}{\partial y}\bigg|_{y=0} \right) =$$

$$= -\nu \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = -\frac{\mu}{\rho}\left.\frac{\partial v_x}{\partial y}\right|_{y=0} = -\frac{1}{\rho}\cdot\tau_w$$

with τ_w = shear stress (viscous component) at the wall ($y=0$). Eq. (11) becomes:

$$-v_\infty\int_0^\infty \frac{\partial v_x}{\partial x}\,dy + \int_0^\infty v_x\frac{\partial v_x}{\partial x}\,dy + \int_0^\infty v_x\frac{\partial v_x}{\partial x}\,dy = -\frac{\tau_w}{\rho}$$

$$\boxed{\frac{\tau_w}{\rho}} = \underbrace{v_\infty\int_0^\infty \frac{\partial v_x}{\partial x}\,dy}_{} - \underbrace{2\int_0^\infty v_x\frac{\partial v_x}{\partial x}\,dy}_{}$$

$$\int_0^\infty v_\infty\frac{\partial v_x}{\partial x}\,dy = \int_0^\infty \frac{\partial(v_x v_\infty)}{\partial x}\,dy \qquad \int_0^\infty 2v_x\frac{\partial v_x}{\partial x}\,dy = \int_0^\infty \frac{\partial v_x^2}{\partial x}\,dy$$

$$= \int_0^\infty \frac{\partial(v_x v_\infty)}{\partial x}\,dy - \int_0^\infty \frac{\partial v_x^2}{\partial x}\,dy$$

$$= \int_0^\infty\left[\frac{\partial}{\partial x}\left(v_x v_\infty - v_x^2\right)\right]dy$$

INVERT $\int$ WITH $\frac{\partial}{\partial x}$

$$= \frac{\partial}{\partial x}\int_0^\infty\left(v_x v_\infty - v_x^2\right)dy = \frac{\partial}{\partial x}\int_0^\infty \frac{v_x}{v_\infty}\left(1-\frac{v_x}{v_\infty}\right)v_\infty^2\,dy$$

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial x}\Bigg[\underbrace{\int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy}_{\text{This quantity is defined as MOMENTUM THICKNESS}}\Bigg] \cdot v_\infty^2$$

$v_\infty = \text{const}$!

$$\boxed{\hat{\delta} \triangleq \int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy}$$ MOMENTUM THICKNESS

$$\Downarrow$$

$$\tau_w = \rho\, v_\infty^2 \frac{\partial \hat{\delta}}{\partial x}$$

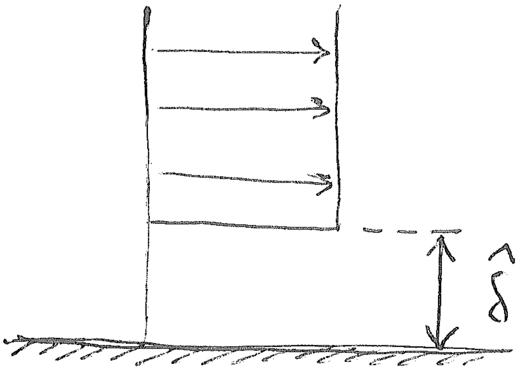
$\hat{\delta} = \hat{\delta}(x) \rightarrow$

$$= \rho\, v_\infty^2 \frac{d\hat{\delta}}{dx} \qquad [12]$$

Physical meaning of the momentum thickness: if there was no B.L. (namely no region of the flow where viscosity matters), then the flow would be potential everywhere and there would be no "need" to satisfy the no-slip condition: the fluid velocity could be equal to v_∞ every= where. In reality, this does not happen and the fluid velocity is smaller than v_∞ inside the B.L. This implies that, compared to an ideal case of negligible viscosity effects, there will be a loss (or, better, reduction) of momentum:

$$\frac{\dot{m}}{w} \triangleq \int_0^\infty \rho v_x (v_\infty - v_x)\, dy \quad \circledast$$

One can imagine a fictitious (not physical) situation in which the velocity profile has the following shape:



$$v_x = \begin{cases} v_\infty & \text{for } y \geq \hat{\delta} \\ 0 & \text{for } y < \hat{\delta} \end{cases}$$

$\hat{\delta}$ is the distance by which the velocity profile should be displaced along y (and away from the plate) to generate the same reduction of momentum as in the real B.L.:

$$\frac{\dot{m}}{w} \triangleq \rho v_\infty^2 \cdot \hat{\delta} \quad \oplus$$

Equalling ⊛ and ◈ one gets:

$$\rho v_\infty^2 \hat{\delta} = \int_0^\infty \rho v_x (v_\infty - v_x)\, dy$$

$$\hat{\delta} = \int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy \qquad \underbrace{\text{Q.E.D.}}_{\text{Quod Erat Demonstrandum}}$$

Now: $\eta = y/\delta(x) \rightarrow y = \eta \cdot \delta(x) \rightarrow \boxed{dy = \delta(x)\, d\eta}$

Hence:

$$\hat{\delta} = \int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy =$$

$$= \int_0^{\delta(x)} \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy + \underbrace{\int_{\delta(x)}^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy}_{=0 \text{ because } v_x = v_\infty \text{ for } y > \delta(x)!}$$

If $y = \delta(x)$ then $\eta = 1$

$$= \int_0^1 \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) \delta(x)\, d\eta$$

and eqn. [12] at page 20 becomes:

$$\tau_w = \rho v_\infty^2 \frac{d}{dx}\left[\delta(x) \underbrace{\int_0^1 \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) d\eta}_{\text{THIS TERM IS INDEPENDENT OF } x}\right]$$

In conclusion:

$$\tau_w = \rho v_\infty^2 \frac{d\delta(x)}{dx} \int_0^1 \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) d\eta$$

This equation can be used to determine the B.L. growth in the turbulent region.

From τ_w one can define the SHEAR VELOCITY:

$$v_\tau \triangleq \sqrt{\frac{\tau_w}{\rho}} \qquad (\tau_w = \rho u_\tau^2)$$

This velocity has no straightforward physical meaning: Close to the wall, τ_w is important and it is used to define a velocity scale that is appropriate in the near-wall region.

Experimental observations and numerical simulations have shown that, in a turbulent B.L.:
(over a flat plate)

$$\boxed{\frac{v_x}{v_\tau} = 8.74\left(\frac{y \cdot v_\tau}{\nu}\right)^{1/7}} \qquad [13]$$

Eq. [13] must be valid also for $y = \delta(x)$:

$$@\ y = \delta(x) \Rightarrow \frac{v_x}{v_\tau} = \frac{v_\infty}{v_\tau} = 8{,}74\left(\frac{\delta(x)\cdot v_\tau}{\nu}\right)^{1/7}$$

$$v_x(y = \delta(x)) = v_\infty$$

One thus gets:

$$\boxed{\frac{v_x}{v_\infty}} = \frac{v_x}{v_\tau}\cdot\frac{1}{\left(\frac{v_\infty}{v_\tau}\right)}$$

$$= \cancel{8{,}74}\left(\frac{y\cdot\cancel{v_\tau}}{\cancel{\nu}}\right)^{1/7}\cdot\frac{1}{\cancel{8{,}74}\left(\frac{\delta(x)\cdot\cancel{v_\tau}}{\cancel{\nu}}\right)^{1/7}}$$

$$= \left(\frac{y}{\delta(x)}\right)^{1/7} = \boxed{\eta^{1/7}} \qquad [14]$$

Going back to the equation for τ_w, eq. [14] yields:

$$\tau_w = \rho\, v_\infty^2\, \frac{d\delta(x)}{dx}\overbrace{\int_0^1 \eta^{1/7}\left(1-\eta^{1/7}\right) d\eta}^{=\,7/72}$$

$$\tau_w = \frac{7}{72}\,\rho\, v_\infty^{\,2}\,\frac{d\delta(x)}{dx} \qquad [15]$$

Combining eq. [13] with $v_\tau = \sqrt{\frac{\tau_w}{\rho}}$ one also gets:

$$\frac{v_\infty}{v_\tau} = 8.74\left(\frac{\delta(x)\cdot v_\tau}{\nu}\right)^{1/7}$$

$$v_\infty = 8.74\left(\frac{\delta(x)}{\nu}\right)^{1/7}\cdot v_\tau^{\,8/7}$$

$v_\tau = \sqrt{\frac{\tau_w}{\rho}} \leadsto$

$$= 8.74\left(\frac{\delta(x)}{\nu}\right)^{1/7}\cdot\left(\frac{\tau_w}{\rho}\right)^{4/7}$$

$$\tau_w = \left[\frac{v_\infty}{8.74\left(\frac{\delta(x)}{\nu}\right)^{1/7}}\right]^{7/4}\cdot\rho$$

$$= \rho\,\underbrace{\left(\frac{1}{8.74}\right)^{7/4}}_{0{,}0225}\cdot\left[\frac{\nu}{\delta(x)}\right]^{1/4}\cdot v_\infty^{\,7/4}$$

$$= 0{,}0225\,\rho\left[\nu/\delta(x)\right]^{1/4}\cdot v_\infty^{\,7/4} \qquad [16]$$

Equalling eq. [15] and eq. [16]:

$$\frac{7}{72}\cancel{\rho} v_\infty^2 \frac{d\delta(x)}{dx} = 0{,}0225\cancel{\rho}\left[\frac{\nu}{\delta(x)}\right]^{1/4} \cdot v_\infty^{7/4}$$

$$\frac{d\delta(x)}{dx} \cong 0{,}23\left[\frac{\nu}{\delta(x)}\right]^{1/4} \cdot v_\infty^{1/4}$$

Upon seperation of variables:

$$\int_{\circ}^{\delta(x)} d\delta(x) \cdot \delta(x)^{1/4} \cong 0{,}23(\nu/v_\infty)^{1/4} \cdot \int_0^x dx$$

$\uparrow$

We assume that the laminar region of the B.L. is short (this is true, actually!) so that the turbulent region starts at $x \simeq 0$, when $\delta(x)$ is still small and one can put $\delta \simeq 0$!

$$\frac{4}{5}\delta(x)^{5/4} \cong 0{,}23\ (\nu/v_\infty)^{1/4} \cdot x$$

$$\delta(x) = \underbrace{\left(\frac{5}{4}\cdot 0{,}23\right)^{4/5}}_{\simeq 0{,}37} \cdot (\nu/v_\infty)^{1/5} x^{4/5}$$

In the turbulent region of the B.L. $\delta(x) \propto x^{4/5}$

Since $\delta(x) = 0{,}37\,(\nu/v_\infty)^{1/5} \cdot x^{4/5}$ one also finds, from eq. [16]:

$$\tau_w = 0{,}0225\,\rho \left[\frac{\nu}{\delta(x)}\right]^{1/4} \cdot v_\infty^{7/4}$$

$$= \frac{0{,}0225\,\rho\,\nu^{1/4}\,v_\infty^{7/4}}{\left[0{,}37\,(\nu/v_\infty)^{1/5}\,x^{4/5}\right]^{1/4}}$$

$$\cong 0{,}0287\,\rho\,v_\infty^2 \left(\frac{\nu}{v_\infty \cdot x}\right)^{1/5}$$

namely $\tau_w \propto x^{-1/5}$ in turbulent conditions

The expressions developed in this section for $\delta(x)$ and τ_w are applicable for values of $Re_x \triangleq \frac{v_\infty \cdot x}{\nu}$ up to about 10^7. For higher values of the Reynolds number, more complicated velocity profiles should be considered.

EXAMPLE OF CALCULATION OF $\delta(x)$ AND τ_w:

$v_\infty = 1$ m/s

$\nu = 1{,}57 \cdot 10^{-5}$ m^2/s (AIR) $\Rightarrow$ $\begin{cases} \delta(x) = 0{,}2555 \text{ m} \\ \quad\;\; \cong 25{,}5 \text{ cm} \\ \tau_w \cong 2{,}58 \cdot 10^{-3} \frac{N}{m^2} \end{cases}$

$x = 10$ m

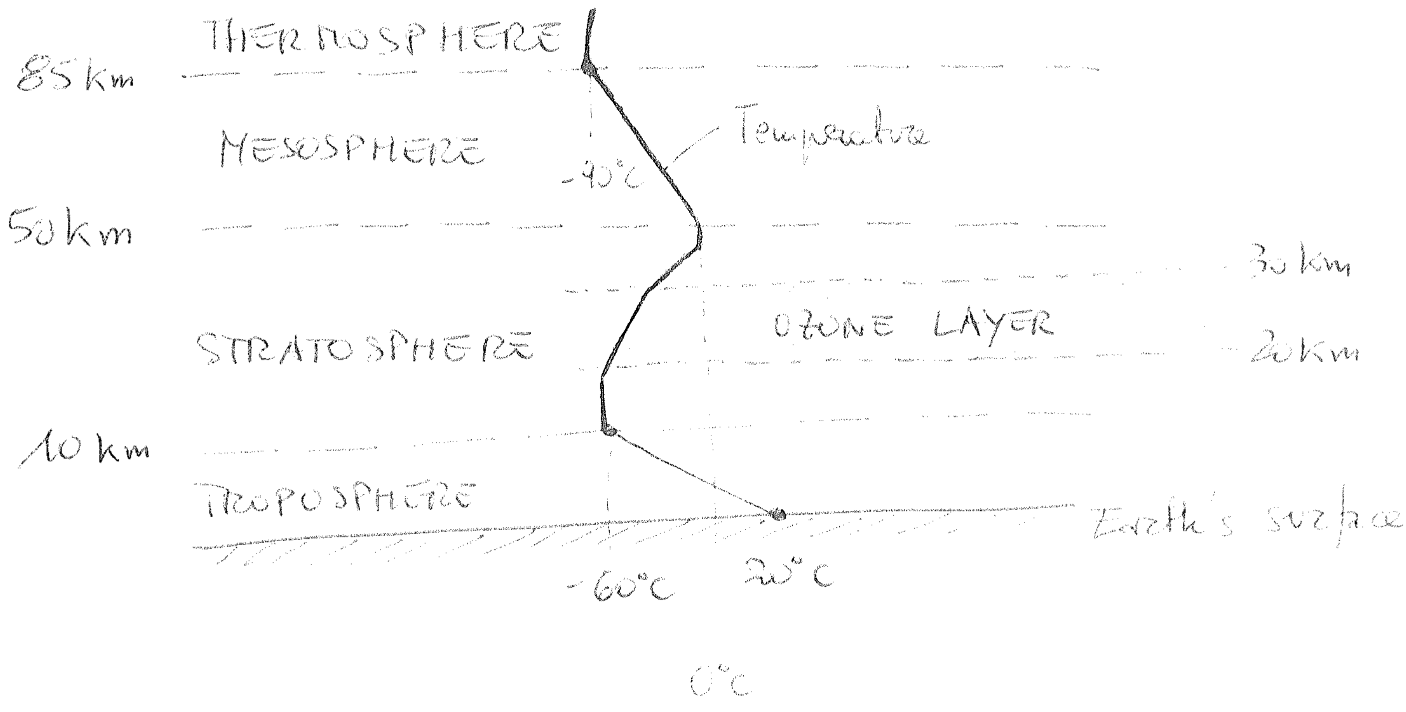
With water ($\nu = 10^{-6}\ m^2/s$) :

$$\delta(x = 10m) = 0,1473\ m \simeq 14,73\ cm$$

$$\tau_w(x = 10m) = 1,14\ N/m^2$$

ATMOSPHERIC BOUNDARY LAYER
(aka PLANETARY BOUNDARY LAYER)

The Earth's atmosphere is more than 100km thick and is typically divided into layers:



The lowest portion of the atmosphere, the troposphere is vital to life on Earth. From an environmental point of view, the troposphere is important because this is where all man-made gaseous emissions are discharged. From a meteorological point of view,