Two-dimensional Vorticity Dynamics

Two-dimensional Vorticity Dynamics

$$\vec{u} = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \\ 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} 0 \\ 0 \\ \omega(x,y,t) \end{pmatrix} \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Consider the stream function $\psi(x,y)$

$$\vec{A} = \psi(x,y)\,\vec{e}_z \qquad \vec{u} = \nabla \times \vec{A}$$

$$\vec{\omega} = -\Delta\psi\,\vec{e}_z$$

Biot and Savart Law: 2D Flows

$$\psi(x,y,t) = -\frac{1}{2\,\pi}\int\int \omega(x',y',t)\,\ln(R)\,dx'\,dy' + \psi_p$$

$$R^2 = (\vec{r}-\vec{r'})^2 = (x-x')^2 + (y-y')^2$$

$$\vec{u} = \frac{1}{2\,\pi}\int\int \frac{\omega(x',y',t)\,\vec{e}_z \times (\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^2}\,dx'\,dy' + \nabla\phi$$

For large r it decreases algebraically $\Gamma/(2\pi r)$

Viscous Flow $\nu \neq 0$

$$\frac{D\omega}{Dt} = \nu \Delta \omega$$

Diffusion of Vorticity

Cancellation of Vorticity

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}$$

A unique vortex layer $\omega = \frac{\Gamma}{\sqrt{4\nu t}} \exp(-\frac{y^2}{4\nu t})$

$$t = 0 \qquad \omega \sim \Gamma \delta(t)$$

Vorticity diffuses by Viscous Effect

Circulation is conserved

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}$$

An array of vortex layers $\omega = \omega_0 \sin(ky)$

$$\omega(y,t) = \omega_0 \exp(-k^2 t) \sin(ky)$$

Vorticity Cancellation by Viscous Effect

The total vorticity field $\nu \neq 0$

$$\Gamma = \int \int \omega \, dS$$

$$\vec{OC} = \frac{\int \int \omega \, \vec{OM} \, dS}{\int \int \omega \, dS}$$

C : Position of the vorticity centroid

$$a^2 \equiv \frac{\int \int \omega \, |\vec{CM}|^2 \, dS}{\int \int \omega \, dS}$$

Size of total vorticity repartition

$$a^2 = a_0^2 + 4 \nu \, t$$

Unique Vortex $\nu \neq 0$

$$\Gamma = \int\int \omega \, dS$$

$$\vec{OC} = \frac{\int\int \omega \, \vec{OM} \, dS}{\int\int \omega \, dS}$$

C : Position of the vortex center the centroid

$$a^2 \equiv \frac{\int\int \omega \, |\vec{CM}|^2 \, dS}{\int\int \omega \, dS}$$

Size of the vortex

$$a^2 = a_0^2 + 4\,\nu\,t$$

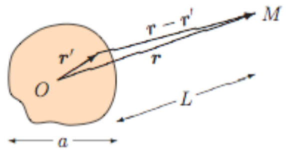
Inviscid vortex flow $\nu = 0$

Conservation of vorticity pointwise

In an assembly of vortices this means :

$\Rightarrow$ Conservation of circulation of any given vortex

$\Rightarrow$ Size of any given vortex
(only for a uniform vorticity)



$$\vec{OC} = \frac{\int\int \omega\, \vec{OM}\, dS}{\int\int \omega\, dS}$$

$$|\vec{r}'| << |\vec{r}|$$

$$\ln(R) = \frac{1}{2}\,\ln|\vec{r}-\vec{r}'|^2 = \frac{1}{2}\,\ln\left[r^2\left(1-\frac{2}{r^2}\,\vec{r}\cdot\vec{r}'\right)\right]$$
$$\sim \ln(r) - \frac{\vec{r}\cdot\vec{r}'}{\vec{r}^2} + \mathcal{O}\left(\frac{a^2}{L^2}\right)$$

$$\psi = -\frac{\ln(r)}{2\,\pi}\underbrace{\int\int \omega\, dx'\, dy'}_{\Gamma} + \frac{\vec{r}}{2\,\pi\, r^2}\underbrace{\int \vec{r}'\,\omega\, dx'\, dy'}_{\vec{r}_C=\vec{0}} + \mathcal{O}\left(\frac{a^2}{r^2}\right)$$

Potential vortex flow

$$\psi \sim \underbrace{-\frac{\Gamma}{2\,\pi}\,\ln(r)}_{\text{point vortex}} + \mathcal{O}\left(\frac{a^2}{r^2}\right)$$

At a point far away from a vortex location,
the velocity field produced by a vortex is similar to a

point-vortex velocity $\omega = \Gamma\delta(x - x_c)\delta(y - y_c)$

Potential vortex flow

$$u_r(r) = \frac{1}{r}\,\frac{\partial\psi}{\partial\theta} = 0, \qquad u_\theta(r) = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\,\pi\,r}$$

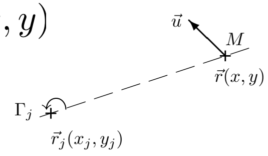
For the velocity field at M, only two quantities matter

The distance OM to the vortex centroid O

The vortex circulation Γ

Velocity due to vortex j at point $\vec{r}(x,y)$

$$\vec{u}_j = \frac{\Gamma}{2\,\pi\,|\vec{r}-\vec{r}_j|^2} \begin{pmatrix} -(y-y_j) \\ x-x_j \\ 0 \end{pmatrix}$$



A system of point vortices : Dynamical system

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \sum_{j\neq i} \frac{\Gamma_j}{2\,\pi\,R_{ij}^2} \begin{pmatrix} -(y_i-y_j) \\ x_i-x_j \end{pmatrix}, \qquad R_{ij}^2 = |\vec{r}_i-\vec{r}_j|^2$$

This system is an Hamiltonian system

$$H = -\frac{1}{4\,\pi}\sum_{k=1}^{N}\sum_{j\neq k}\Gamma_k\,\Gamma_j\,\ln(R_{kj})$$

Interaction energy

Kinetic energy is infinite

$$q_i \equiv x_i \qquad p_i \equiv \Gamma_i\,y_i$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

H is a conserved energy

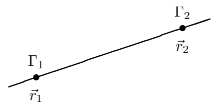
Other Conserved Quantities

A centroid

$$\begin{pmatrix} x_C \\ y_C \end{pmatrix} = \frac{1}{\Gamma} \sum_{i=1}^{N} \Gamma_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

A dispersion radius

$$d^2 = \frac{1}{\Gamma} \sum_{i=1}^{N} \Gamma_i \left[(x_i - x_C)^2 + (y_i - y_C)^2 \right]$$



System of Two point vortices

4 dynamical variables x_1, y_1, x_2, y_2

3 Conserved quantities

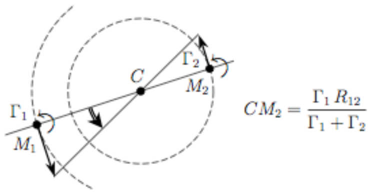
$$H = -\frac{1}{2\,\pi}\,\Gamma_1\,\Gamma_2\,\ln(R_{12}) = Cst \quad \Rightarrow \quad R_{12} = |\vec{r}_1 - \vec{r}_2| = Cst$$

The dispersion radius gives again $R_{12} = |\vec{r}_1 - \vec{r}_2| = Cst$

$$\Gamma_1\,y_1 + \Gamma_2\,y_2 = cte \qquad \Gamma_1\,x_1 + \Gamma_2\,x_2 = cte$$

Two point vortices $\Gamma_1 \Gamma_2 > 0$

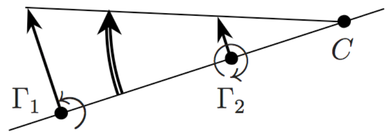
C is located between 1 and 2.



The system rotates around C at angular velocity

$$\Omega = \frac{\Gamma_1}{2\pi R_{12}} \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 R_{12}} = \frac{\Gamma_1 + \Gamma_2}{2\pi R_{12}^2}$$

Two point vortices $\Gamma_1\Gamma_2 < 0$



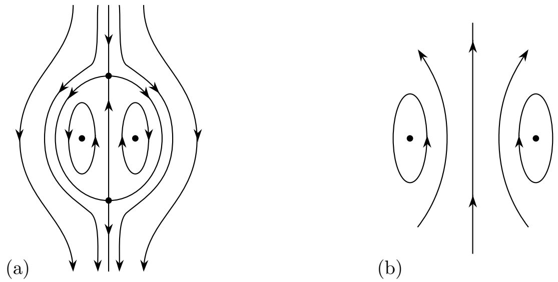
C is located on the side of the more intense vortex

Two Point Vortices: a Dipole $\Gamma_1 + \Gamma_2 = 0$

The dipole does not rotate, and translates at velocity

$$\Gamma/(2\pi\, R_{12})$$

streamlines in the comoving Frame   streamlines in the Laboratory



Trapping and Transporting Fluid

Three Point Vortices

Simple dynamics : an integrable motion

Triangle shape oscillates with a period T1

Triangle rotates with a period T2

$\frac{T_1}{T_2}$ is rational number (initial condition) $\Rightarrow$ periodic

$\frac{T_1}{T_2}$ is in general not in an irrational number

$\Rightarrow$ Quasi-periodic Motion

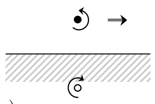
Four or more point Vortices

8 dynamical variables

4 Conserved quantities

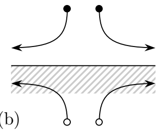
The dynamical system is non integrable: chaotic

Point Vortices with a Wall



(a)

A unique vortex along a wall

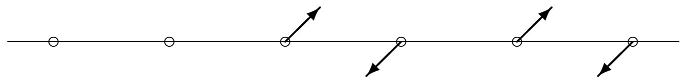


(b)

A Dipole along a wall

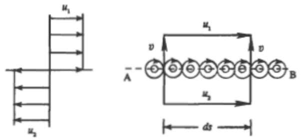
Image theory

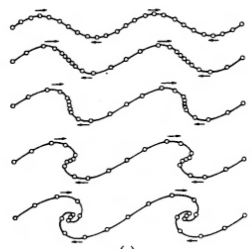
Infinite Row of Point Vortices



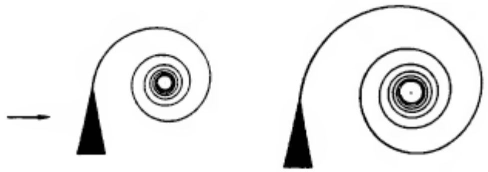
Pairing instability

From an Infinite Row of Point Vortices towards Vortex Sheet

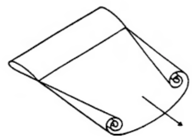




Vortex Sheet

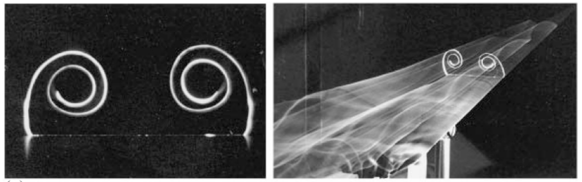


self-similar spiralling

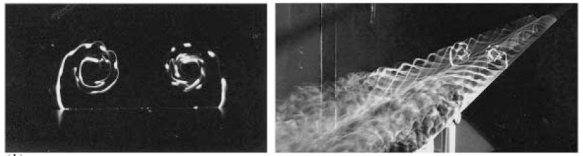


Roll-up behind a wing

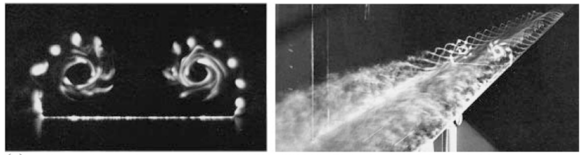
Vortex sheet near trailing edges of a Delta wing



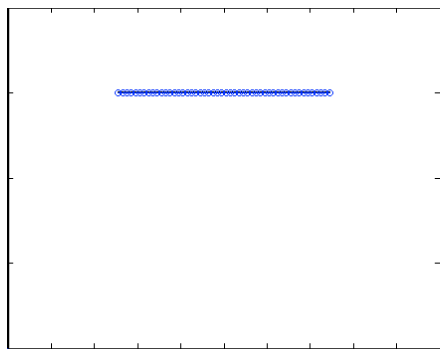
(*a*) $Re_x = 25\,200$, $Re_c = 41\,900$



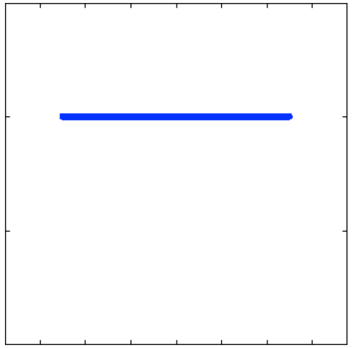
(*b*) $Re_x = 61\,100$, $Re_c = 101\,900$



(*c*) $Re_x = 118\,000$, $Re_c = 196\,700$



A First Vortex sheet with zero total circulation

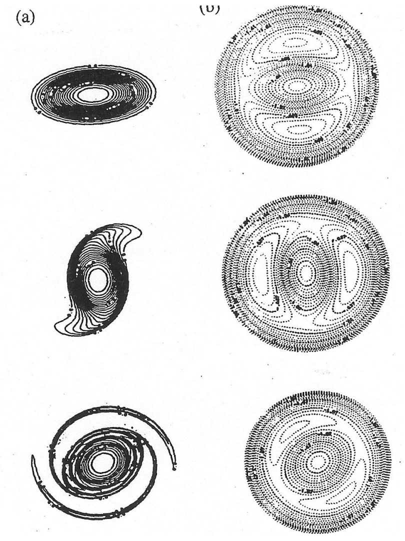


Another Vortex sheet with non nul vortex circulation

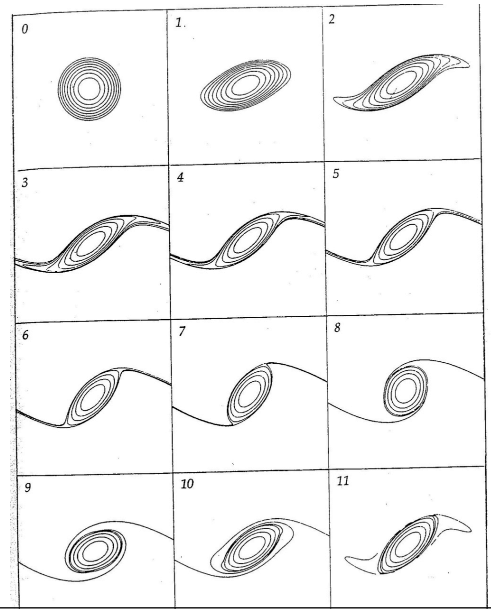
EXTENDED 2D VORTICES

Vortex axisymmetrization

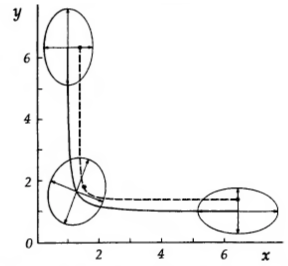
Vorticity   Streamfunction



Vortex in a Potential Strain field

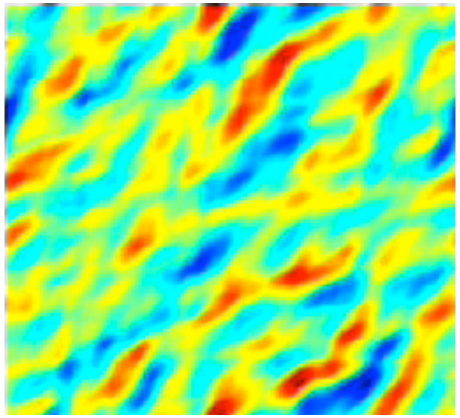


Stripping



Vortex along a wall or a Dipole

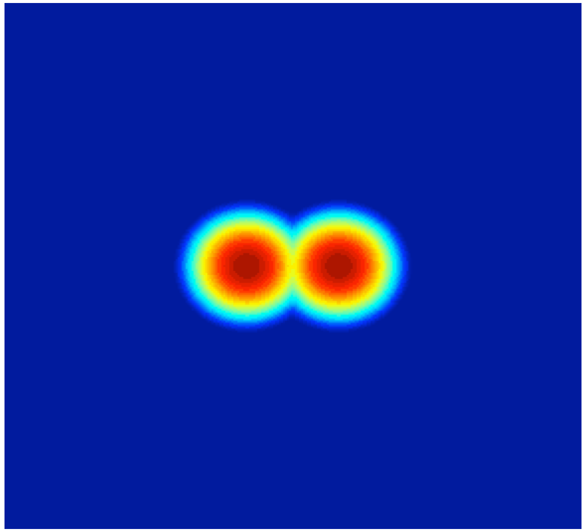
Merging of Vortices



Two-dimensional Turbulence : Inverse Cascade

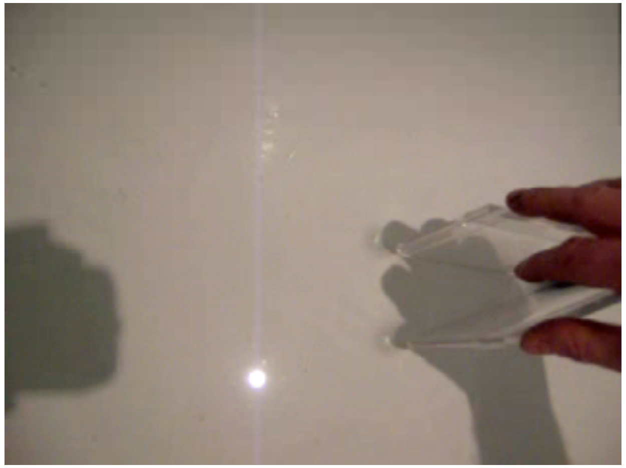
Gerris (S.Popinet)

Vortex Merging



Josserand and Rossi 2005

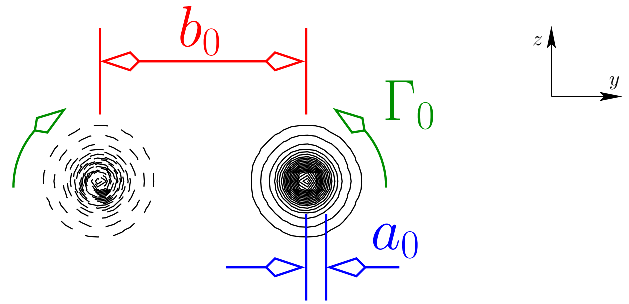
Vortex Dipole

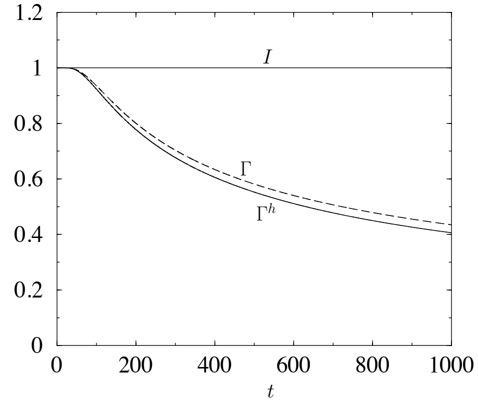


Vortex In a bath

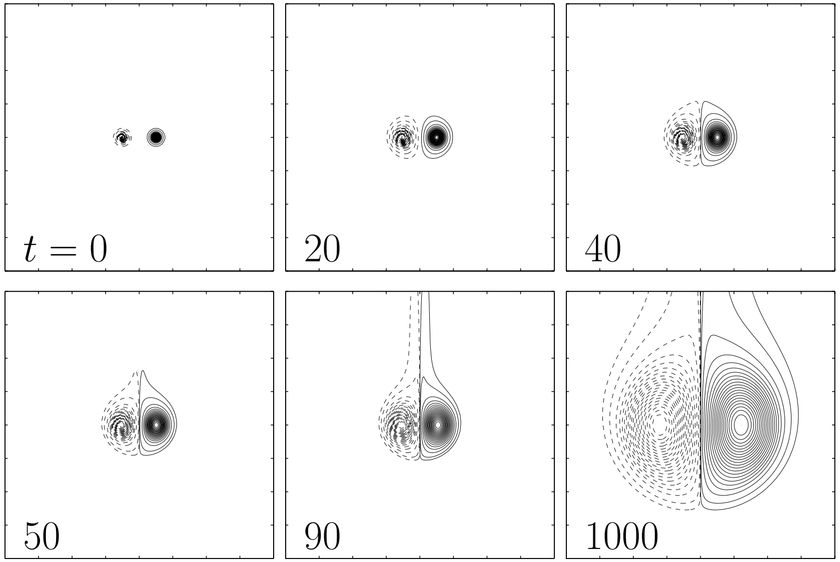
$$\omega(y,z,t=0) = \omega^{(+)}(y,z) + \omega^{(-)}(y,z)$$

$$\omega^{(\pm)}(y,z) = \pm\frac{\Gamma_0}{\pi a_0^2}\exp\left(-\frac{(y \mp b_0/2)^2 + z^2}{a_0^2}\right)$$

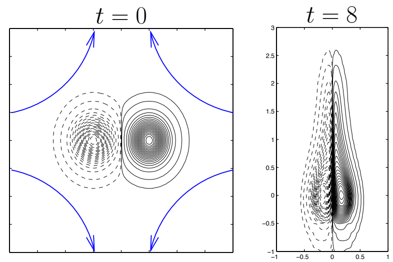




Cancellation of Vorticity



$t = 0$
20
40
50
90
1000



Strain dipole in the Plane

Numerical Simulation   Shallow Water equations
(Code Gerris S.Popinet)