

Global Invariants of Vorticity Dynamics

$$\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times \vec{u}) = -\nu \nabla \times (\nabla \times \vec{\omega})$$

In 3D $\vec{P} = \frac{\rho}{2} \int \vec{x} \times \omega d\tau$ global momentum

$$I = -\frac{\rho}{2} \int \vec{x}^2 \omega d\tau$$
 global angular momentum

When $\nu = 0$ $H = \int \vec{u} \cdot \vec{\omega} d\tau$ helicity

Interpretation of P as Global momentum

One starts with rest state $\vec{u} = \vec{0}$

Let us apply external forces.

They are introduced in the Navier-Stokes equations through a density of force per unit mass $\vec{F}_{ext}(\vec{x}, t)$

$$\begin{cases} \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla p + \vec{F}_{ext}(\vec{x}, t) + \nu \Delta \vec{u} \\ \nabla \cdot \vec{u} = 0 \end{cases}$$

$$\vec{P} = \int_0^t \int_{v_\infty} \rho \vec{F}_{ext}(\vec{x}, t) d^3x \quad \text{Global momentum}$$

Let us consider an impulsive force:

$$\vec{F}_{ext}(\vec{x}, t) = \vec{F}^{(0)}(\vec{x}) \delta(t)$$

Dirac function

$$\vec{P} = \rho \int_{v_\infty} d^3x \int_{0^-}^{0^+} \vec{F}^{(0)}(\vec{x}) \delta(t) dt = \rho \int_{v_\infty} \vec{F}^{(0)}(\vec{x}) d^3x$$

$$0^+ \equiv 0 + \Delta t/2$$

Let \vec{a} be vector field in a three-dimensional space

$$\int_v \vec{x} \times [\nabla \times \vec{a}] d\tau = (\mathcal{N} - 1) \int_v \vec{a} d\tau + \oint_S \vec{x} \times (\vec{n} \times \vec{a}) dS \quad \mathcal{N} = 3$$

$$\rho \int_v \vec{F}^{(0)} d\tau = \frac{\rho}{2} \int_v \vec{x} \times [\nabla \times \vec{F}^{(0)}] d\tau$$

Integrating Navier-Stokes around $t=0$

$$\begin{cases} \vec{u}(0^+) = -\frac{1}{\rho} \nabla \left[\int_{0^-}^{0^+} p dt \right] + \vec{F}^{(0)}(\vec{x}) \\ \nabla \cdot \vec{u}(0^+) = 0 \end{cases}$$

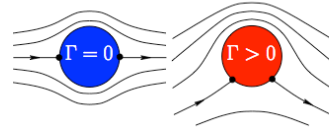
One generates the force to get the appropriate velocity $\vec{u}(0^+)$

$$\vec{\omega}(0^+) = \nabla \times \vec{u}(0^+) = \nabla \times \vec{F}^{(0)}(\vec{x})$$

$$\vec{P} = \frac{\rho}{2} \int \vec{x} \times \omega d\tau$$

Kutta condition

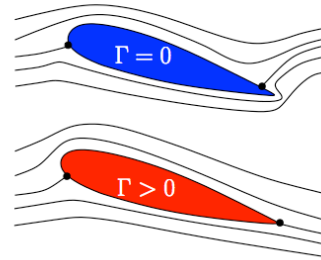
- Circulation around a cylinder



$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l}$$

↪ Circulation Γ is arbitrary

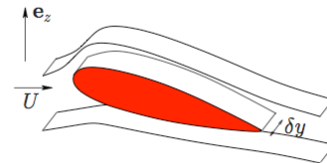
- Circulation around an airfoil



↪ Circulation is determined by the **Kutta condition**.

Kutta-Joukowski theorem

- Lift of a wing element



Kutta-Joukowski theorem :

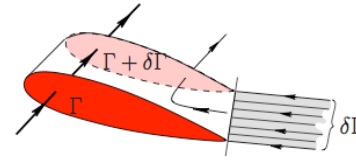
$$L' = \rho U \Gamma$$

$$\delta \mathbf{L} = L' \delta y \mathbf{e}_z$$

$-\delta \mathbf{L}$ is the force exerted by the wing element on the fluid (transfer of vertical momentum).

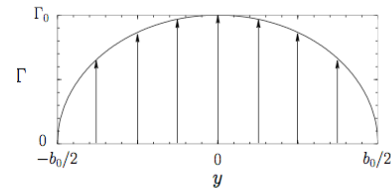
Vorticity sheet of strength :

$$\Omega(y) = -\frac{d\Gamma}{dy}\delta(z)\mathbf{e}_x$$



• Circulation distribution

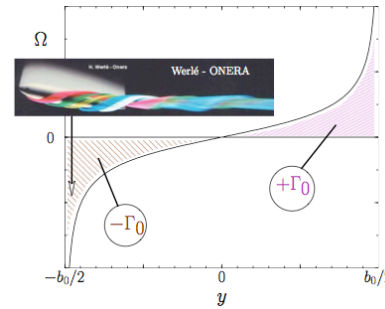
$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{y}{b_0/2}\right)^2}$$

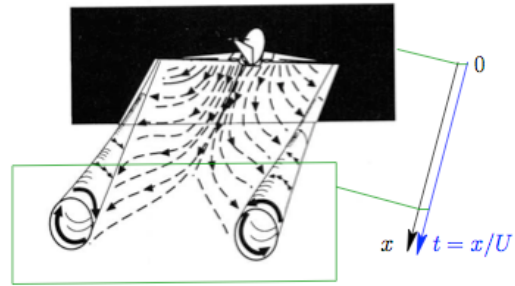


$$L = \int_{-b_0/2}^{b_0/2} \rho U \Gamma(y) dy = \frac{\pi}{4} \rho b_0 U \Gamma_0$$

• Axial vorticity distribution

$$\Omega(y) = \Gamma_0 \frac{y}{\sqrt{1 - \left(\frac{y}{b_0/2}\right)^2}}$$





The momentum is conserved into the two vortices

Conservation of helicity for Euler Flow

$\mathcal{V}(t)$ Region containing vorticity

$$\mathcal{H} \equiv \int_{\mathcal{V}} \boldsymbol{\omega} \cdot \mathbf{u} \, d\tau$$

$$\frac{D\mathcal{H}}{Dt} \equiv \int_{\mathcal{V}} \frac{D[\boldsymbol{\omega} \cdot \mathbf{u}]}{Dt} \, d\tau = \int_{\mathcal{V}} \left(\frac{D\boldsymbol{\omega}}{Dt} \cdot \mathbf{u} + \frac{D\mathbf{u}}{Dt} \cdot \boldsymbol{\omega} \right) \, d\tau$$

$$\frac{D\vec{\omega}}{Dt} = 0 \quad \Rightarrow$$

$$\frac{D\vec{u}}{Dt} = -\nabla[P/\rho]$$

$$\frac{D\mathcal{H}}{Dt} = -\frac{1}{\rho} \int_{\mathcal{V}} \boldsymbol{\omega} \cdot \nabla P \, d\tau = -\frac{1}{\rho} \int_{\mathcal{V}} \nabla \cdot [\boldsymbol{\omega} P] \, d\tau = -\frac{1}{\rho} \int_{S_{\infty}} P \boldsymbol{\omega} \cdot \mathbf{n} \, dS = 0$$

Topological Interpretation of helicity

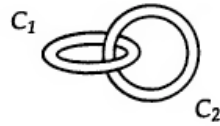
An axisymmetric Ring

$$\vec{u} = u_r(r, z, t) \vec{e}_r + u_z(r, z, t) \vec{e}_z \quad \vec{\omega} = \omega_\theta(r, z, t) \vec{e}_\theta$$

$$\vec{\omega} \cdot \vec{u} = 0 \quad \mathcal{H} \equiv \int_{\mathcal{V}} \boldsymbol{\omega} \cdot \boldsymbol{u} \, d\tau = 0$$

Topological Interpretation of helicity

Two axisymmetric linked rings



$$\vec{\omega}(\vec{x}) = \vec{\omega}_1(\vec{x}) + \vec{\omega}_2(\vec{x})$$

$$\vec{u}(\vec{x}) = \vec{u}_1(\vec{x}) + \vec{u}_2(\vec{x})$$

$$\mathcal{H} \equiv \int_{\mathcal{V}} \boldsymbol{\omega} \cdot \mathbf{u} \, d\tau = \int_{\mathcal{V}_1} \boldsymbol{\omega}_1 \cdot \mathbf{u}_1 \, d\tau + \int_{\mathcal{V}_2} \boldsymbol{\omega}_2 \cdot \mathbf{u}_2 \, d\tau + \int_{\mathcal{V}_1} \boldsymbol{\omega}_1 \cdot \mathbf{u}_2 \, d\tau + \int_{\mathcal{V}_2} \boldsymbol{\omega}_2 \cdot \mathbf{u}_1 \, d\tau$$

$$\int_{\mathcal{V}_1} \boldsymbol{\omega}_1 \cdot \mathbf{u}_1 \, d\tau = \int_{\mathcal{V}_2} \boldsymbol{\omega}_2 \cdot \mathbf{u}_2 \, d\tau = 0 \quad \text{Axisymmetric Ring}$$

$$\mathcal{H} \equiv \int_{\mathcal{V}} \boldsymbol{\omega} \cdot \mathbf{u} \, d\tau = \int_{\mathcal{V}_1} \boldsymbol{\omega}_1 \cdot \mathbf{u}_2 \, d\tau + \int_{\mathcal{V}_2} \boldsymbol{\omega}_2 \cdot \mathbf{u}_1 \, d\tau$$

$$\int_{\mathcal{V}_1} \boldsymbol{\omega}_2 \cdot \mathbf{u}_1 \, d\tau = \int_{C_2} \int_{\Sigma(x_2)} \boldsymbol{\omega}_2 \cdot dS \mathbf{u}_1 \, dx_2$$

Small cross-section $\int_{\Sigma(x_2)} \boldsymbol{\omega}_2 \cdot d\mathbf{S} = \Gamma_2 \vec{e}_2$

$$\int_{V_1} \boldsymbol{\omega}_2 \cdot \mathbf{u}_1 \, d\tau = \Gamma_2 \int_{C_2} \mathbf{u}_1 \cdot \mathbf{e}_2 dx_2$$

$$\int_{C_2} \mathbf{u}_1 \cdot \mathbf{e}_2 dx_2 = \int_{S_2} \boldsymbol{\omega}_1 \cdot \mathbf{n} \, ds$$

If the two rings are unlinked $\int_{S_2} \boldsymbol{\omega}_1 \cdot \mathbf{n} \, ds = 0 \quad \mathcal{H} = 0$

If the two rings are linked $\int_{S_2} \boldsymbol{\omega}_1 \cdot \mathbf{n} \, ds = \pm \Gamma_1 \quad \mathcal{H} = \pm \Gamma_1 \Gamma_2$

Helicity has to do with the number of
knots : topological invariant