Vorticity and Dynamics

In Navier-Stokes equation

Nonlinear term

$\omega \times \vec{u}$ the Lamb vector is related to the nonlinear term

$$(\vec{u} \cdot \nabla)\, \vec{u} = \nabla \left(\frac{\vec{u}^2}{2} \right) + \omega \times \vec{u}$$

Sort of Coriolis force in a rotation frame

Viscous term

$$\nu\, \Delta \vec{u} = \nu \left(\nabla(\nabla.\vec{u}) - \nabla \times (\nabla \times \vec{u}) \right) = -\nu\, \nabla \times \omega$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} + \nabla \left(\frac{\vec{u}^2}{2} \right) = -\nabla [\frac{p}{\rho}] - \nu \nabla \times \vec{\omega}$$

taking the rotational

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{u}) = -\nu \Delta \vec{\omega}$$

Dynamical equation for Vorticity

Pressure is eliminated in the equation

$$\Delta p = 2\,\rho\,Q \qquad\qquad Q = \frac{1}{2}\,[\frac{1}{2}\,\omega^2 - e^2]$$

$Q > 0$ **Q-Criterium** to define a vortex region

Mixing layer

$$\vec{u} = u_x(y)\,\vec{e}_x; \qquad \omega = -\frac{\partial u}{\partial y}$$

$$e_{yx} = e_{xy} = \frac{1}{2}\,\frac{\partial u}{\partial y} \qquad \Rightarrow \qquad \omega^2 = 2\,e_{ij}\,e_{ij} \qquad Q = 0$$

$$e_{xx} = e_{yy} = 0$$

$\Rightarrow$ pressure is uniform

Vortex

$$\vec{u} = u_\theta(r)\,\vec{e}_\theta; \qquad \omega_z = \frac{1}{r}\,\frac{\partial(r u_\theta)}{\partial r}$$

$$e_{rr} = 0, \quad e_{\theta\theta} = 0, \quad e_{r\theta} = \frac{r}{2}\,\frac{\partial}{\partial r}(\frac{u_\theta}{r})$$

$$\Delta p = 2\,\rho\,Q \qquad\qquad Q = \frac{1}{2r}\frac{\partial u_\theta^2}{\partial r}$$

Pressure can be computed using the first component of the Navier-Stokes equation in polar coordinates

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad\qquad p(r) = p(\infty) - \rho\int_r^\infty \frac{u_\theta^2}{r}dr$$

pressure decreases as we get near the vortex center :
it is a pressure low.

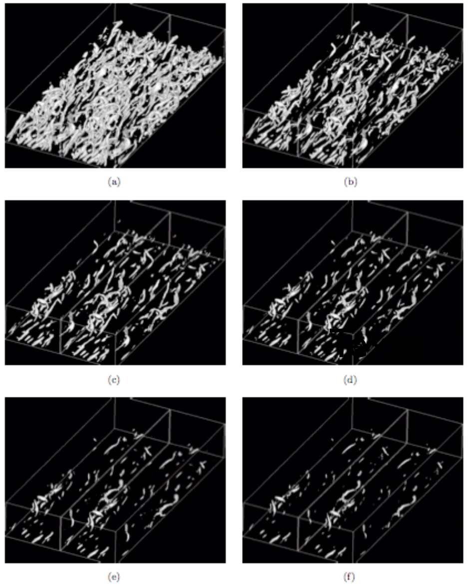
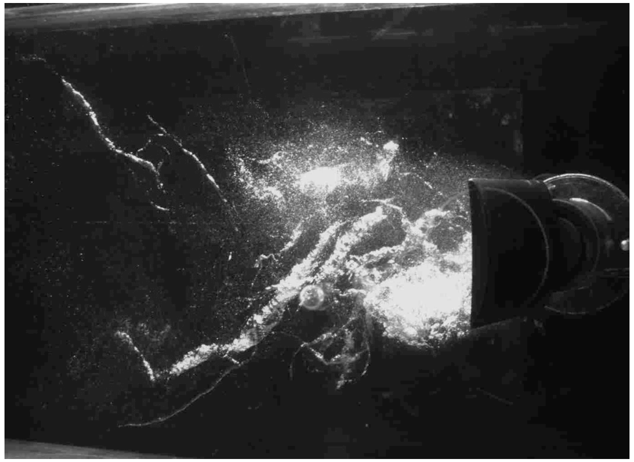


Figure 6. Threshold effects on near-wall vortices educed by the Q criterion: (a) $Q = 0.2$, (b) $Q = 0.4$, (c) $Q = 0.6$, (d) $Q = 0.8$, (e) $Q = 1.0$ and (f) $Q = 1.2$.

Q-vortex
from
Delcayre (2000)



Cavitation (Cadot et al)

Dissipation and Vorticity

Internal dissipation Φ per unit volume

particle deformation → internal energy

$\Phi = 2\,\mu\, e_{ij}\, e_{ij} > 0$ incompressibility assumed

$$\Phi = \mu\,\omega^2 + 2\,\mu\,\nabla.\left[\omega \times \vec{u} + 12\,\nabla(\vec{u}^2)\right]$$

Dissipation and Vorticity

$$\int_v \Phi \, d\tau = \mu \int_v \omega^2 \, d\tau + \mu \int_{\text{surface}} [2\,\omega \times \vec{u} + \nabla(\vec{u}^2)] \cdot \vec{n} \, dS$$

the dissipation per unit mass used in turbulence

$$\epsilon = \frac{1}{\rho\,V} \int_v \Phi \, d\tau = \frac{\nu}{V} \int_v \omega^2 \, d\tau$$

There exists of a zone of dissipation around a vortex

Chaoticity and Vorticity

$$\vec{u} = \vec{u}_{BS} + \nabla\phi$$

Potential only depends on the boundary conditions since it satisfies at each time

$$\Delta\phi = 0 \qquad \vec{n}\cdot\nabla\phi = (\vec{u} - \vec{u}_{BS})\cdot\vec{n}$$

Past history does not play a role for ϕ

Vorticity precisely brings this historical aspect.

Vorticity ω modifies velocity $\mathbf{u}$

Velocity $\mathbf{u}$ modifies with vorticity ω

This implies feedback loop

⇒ Chaoticity and vortex

⇒ Turbulence and vortex

Transport Equation for Vorticity : Incompressible Newtonian Fluid

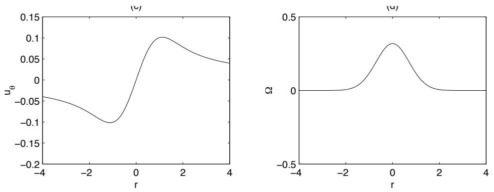
$$\nabla \times [\vec{\omega} \times \vec{u}] = \ (u \cdot \nabla)\vec{\omega} - (\omega \cdot \nabla)\vec{u}$$

$$\underset{\substack{\uparrow \\ \text{Lagrangian Transport}}}{\frac{D\omega}{Dt}} = [\partial_t + \mathbf{u}.\nabla]\omega = \omega.\underset{\substack{\uparrow \\ \text{Stretching}}}{\nabla\mathbf{u}} + \underset{\substack{\uparrow \\ \text{Viscous Diffusion} \\ \downarrow}}{\nu\boldsymbol{\Delta}\omega}$$

Circulation theorem

$$\frac{d\Gamma}{dt} = -\nu \oint_C \nabla \times \omega.d\mathbf{l}$$

Vorticity Diffusion



Lamb-Oseen Vortex

$$U_\theta(r,t) = \frac{\Gamma}{2\pi r}[1 - \exp(-\frac{r^2}{a^2})] \qquad \Omega(r,t) = \frac{\Gamma}{\pi a^2}\exp(-\frac{r^2}{a^2})$$

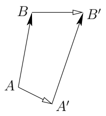
$$a^2(t) = a^2(0) + 4\nu t$$

Inviscid incompressible flows

$$\frac{D\omega}{Dt} = [\partial_t + \mathbf{u}.\nabla]\omega = \omega.\nabla\mathbf{u}$$

Consider a vector element of a material line $\vec{A}\vec{B} = \delta\vec{l}$

During time dt, points A and B move respectively to points A' and B'



$$\vec{A'}\vec{B'} = \delta\vec{l'} = \delta\vec{l} + (\vec{u}_B - \vec{u}_A)\delta t$$

$$\frac{D\delta\vec{l}}{Dt} = \frac{\delta\vec{l'} - \delta\vec{l}}{\delta t} \quad \text{and} \quad \vec{u}_B - \vec{u}_A = (\vec{A}\vec{B}\cdot)\vec{u}$$

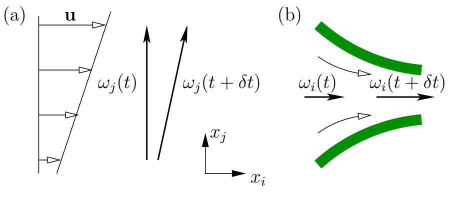
$$\frac{D\delta\vec{l}}{Dt} = (\delta\vec{l} \cdot \nabla)\vec{u}$$

Inviscid incompressible flows

By comparing equations

$$\frac{D\omega}{Dt} = \omega.\nabla\mathbf{u} \qquad \frac{D\delta\vec{l}}{Dt} = (\delta\vec{l}\cdot\nabla)\vec{u}$$

In the inviscid fluid, for a given fluid particle, vorticity evolves in time the same way as does a small material line with the same direction.



it may be tilted

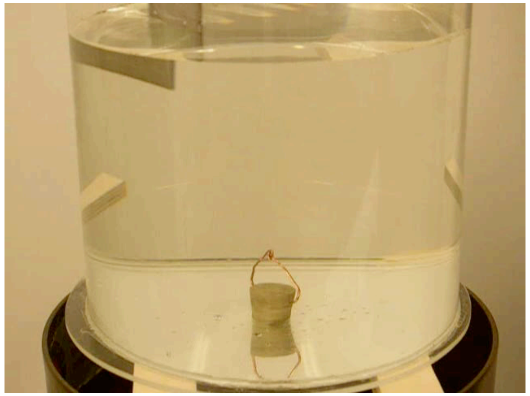
it may be stretched/compressed

Concept used in numerical vortex methods.

Vorticity Enhancement by Strain Field

$$\frac{D\omega}{Dt} = [\partial_t + \mathbf{u}.\nabla]\omega = \omega.\nabla\mathbf{u} + \nu\boldsymbol{\Delta}\omega$$

Stretching

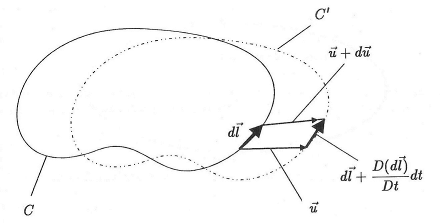


Law of conservation :
Helmholtz Laws for inviscid incompressible flows

Hermann von Helmholtz (1821-1894) during his stay in Washington, D.C., September 1893 (photograph by M. Brady, courtesy Inter Nationes e.V., Bonn, Germany).

First Law of conservation

Circulation around a material loop



$$\Gamma = \oint_C \mathbf{u}.d\mathbf{l} = \int_S \omega.d\mathbf{S}$$

$$\frac{d}{dt}\int_{C(t)} \mathbf{u}\cdot\mathbf{dx} = \int_{C(t)} \frac{D\mathbf{u}}{Dt}\cdot\mathbf{dx}$$

$$\frac{d}{dt}\Gamma = -\nu\int_{C(t)} \boldsymbol{\nabla}\times\boldsymbol{\omega}\cdot\mathbf{dx}$$

$$\nu = 0$$

$$\frac{d}{dt}\Gamma = 0$$

Circulation along a material line remains constant when convected by the fluid

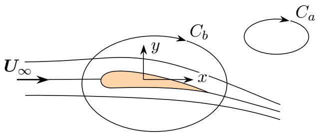
Second Law of conservation

Consider a fluid particle without vorticity

All small loop around the particle would have zero circulation

$$\frac{d}{dt}\Gamma = 0 \qquad \Rightarrow \quad \text{all small loop remain of zero circulation}$$

$$\Rightarrow \quad \omega = 0$$



A potential region remains potential
when convected by the fluid in the inviscid context

Third Law of conservation.

Vortex lines are material lines i.e. convected by the fluid

↓

Tilting and Stretching of Vortex lines

First and Third laws imply that:

Circulation of a vortex tube does not change with time

Helmholtz conservation laws in 2D

For the two-dimensional case u(x,y), v(x,y), w=0

$$\omega = \omega\, \vec{e}_z = \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \right) \vec{e}_z$$

Vorticity lines are straight and there is no stretching

$$(\vec{u} \cdot \nabla)\, \omega = 0$$

In 2D Euler, vorticity is such that $\frac{D\omega}{Dt} = 0$

$\Rightarrow$ Vorticity of each particle is hence conserved

Helmholtz conservation laws in axisymmetric case

axisymmetric vortex ring

$$\vec{u} = u_r(r,z,t)\,\vec{e}_r + u_z(r,z,t)\,\vec{e}_z$$

$$\omega = \omega_\theta(r,z)\,\vec{e}_\theta$$

$$\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}$$

Vorticity lines are rings and there is stretching

In Euler, vorticity is such that $\frac{D}{Dt}\left(\frac{\omega_\theta}{r}\right) = 0$

$\Rightarrow \quad \frac{\omega_\theta}{r}$ is hence conserved