Biot and Savart Law (3D)

"inverting" vorticity definition

$$\mathbf{U}(\mathbf{x},t) = \frac{1}{4\pi}\int \frac{\mathbf{\Omega}(\mathbf{y},t)\times(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} d^3\mathbf{y} + \nabla\phi$$

Almost a one to one relation between
Velocity and vorticity field

$\nabla\phi$ Allows to satisfy the normal boundary conditions for the Velocity field

When the domain is infinite and the velocity field finite $\nabla\phi = 0$

Consider an instantaneous vorticity field localized in 3D space.

Because Incompressibility $\nabla.\vec{u} = 0$

Assume $\vec{A}$ to be the potential vector of incompressible velocity field

$$\vec{u} = \nabla \times \vec{A}$$

$$\omega = \nabla \times \vec{u} = \nabla \times [\nabla \times \vec{A}] = \nabla(\nabla.\vec{A}) - \Delta\vec{A}$$

If one assumes $\nabla.\vec{A} = 0$

Poisson equation $\Delta A_i + \omega_i = 0$

$$A_i(\vec{x}) = \frac{1}{4\,\pi} \int \frac{\omega_i(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3\vec{x}'$$

Velocity field away from the 3D Vorticity Domain.

Consider a compact 3D vorticity field

$$|\vec{x} - \vec{x}'|^2 = \vec{x}^2 + \vec{x}'^2 - 2\,\vec{x}\cdot\vec{x}'$$

$$\vec{A}(\vec{x}) = -\mu \times \nabla \left(\frac{1}{\vec{x}}\right) \qquad \mu \equiv \frac{1}{8\,\pi} \int \vec{x}' \times \omega(\vec{x}')\, d\tau'$$

is a dipolar field like in magnetostatics

$$\vec{u} \longleftrightarrow \vec{B}$$
$$\vec{A} \longleftrightarrow \vec{A}$$
$$\omega \longleftrightarrow \vec{j}$$

flow is potential

$$\vec{u} = \nabla\left[\Phi\right] \qquad \Phi \equiv (\mu \cdot \nabla)\,\frac{1}{r}$$

μ is related to the impulsion

$\vec{P} = 4\,\pi\,\rho\,\mu$ which is a global invariant

Away from a turbulent flow, the velocity is steady !

Biot and Savart Law: 3D Flows

For large r it decreases algebraically $1/r^3$

Biot and Savart Law: 2D Flows

For large r it decreases algebraically $\Gamma/(2\pi r)$

Biot and Savart law is valid:

Discontinuous behaviour of ω : vorticity jump

ω vorticity discontinuous

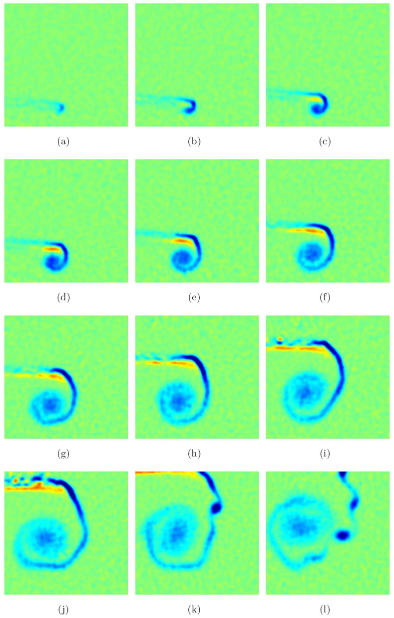
$\vec{u}$ velocity continuous but derivative discontinuous

Singular behaviour of ω

ω vorticity infinite

$\vec{u}$ velocity discontinuous

point vortex or Vorticity sheet



Roll-up of vortex sheets

Experimental
Vortex “sheet”
(Leweke et al 2007)