Transport of TKE and Turbulence Models

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May 29, 2020

Summary I

Transport equation of the turbulent kinetic energy

$k - \mathcal{E}$ model

Alternative turbulence models

Transport equation of the turbulent kinetic energy

Turbulent kinetic energy

The **Turbulent Kinetic Energy** measures the intensity of turbulence and is defined as:

$$\boxed{\mathsf{TKE} = k := \frac{1}{2}\rho\overline{v_i'\,v_i'} = \frac{1}{2}\rho\left[\overline{(v_x')^2 + (v_y')^2 + (v_z')^2}\right]}$$

expressed per unit volume in a Cartesian reference system. It is equivalent to the trace of the Reynolds' stress tensor:

$$\boxed{k = \frac{1}{2}\rho \cdot \mathrm{Tr}\left(\overline{v_i'\,v_j'}\right)}$$

Let us derive the transport equation for TKE.

Derivation of TKE equation

Step 1: Start from NS and RANS

$$NS: \ \rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial {x_j}^2}$$

$$RANS: \ \rho\left(\frac{\partial \overline{v}_i}{\partial t} + \overline{v_j}\frac{\partial \overline{v}_i}{\partial x_j}\right) = -\frac{\partial \overline{P}}{\partial x_i} + \mu\frac{\partial^2 \overline{v}_i}{\partial {x_j}^2} - \rho\frac{\partial \overline{v_i' v_j'}}{\partial x_j}$$

Step 2: Subtract RANS to NS and obtain an eqn. for $v_i' = v_i - \overline{v}_i$

$$\rho\left(\frac{\partial v_i'}{\partial t} + \underbrace{v_j\frac{\partial v_i}{\partial x_j} - \overline{v}_j\frac{\partial \overline{v}_i}{\partial x_j}}_{\boxed{\star}}\right) = -\frac{\partial P'}{\partial x_i} + \mu\frac{\partial^2 v_i'}{\partial {x_j}^2} + \rho\frac{\partial \overline{v_i' v_j'}}{\partial x_j}$$

$$\boxed{\star} = \cancel{\overline{v}_j\frac{\partial \overline{v}_i}{\partial x_j}} + v_j'\frac{\partial \overline{v}_i}{\partial x_j} + \overline{v}_j\frac{\partial v_i'}{\partial x_j} + v_j'\frac{\partial v_i'}{\partial x_j} - \cancel{\overline{v}_j\frac{\partial \overline{v}_i}{\partial x_j}}$$

Derivation of TKE equation

This yields:

$$\rho\left(\frac{\partial v_i'}{\partial t} + v_j'\frac{\partial \overline{v}_i}{\partial x_j} + \overline{v}_j\frac{\partial v_i'}{\partial x_j} + v_j'\frac{\partial v_i'}{\partial x_j}\right) = -\frac{\partial P'}{\partial x_i} + \mu\frac{\partial^2 v_i'}{\partial {x_j}^2} + \rho\frac{\partial \overline{v_i' v_j'}}{\partial x_j}$$

Step 3: Multiply by v_i'

$$\rho\left(\underbrace{v_i'\frac{\partial v_i'}{\partial t}}_{[1]} + v_i' v_j'\frac{\partial \overline{v}_i}{\partial x_j} + \underbrace{v_i'\overline{v}_j\frac{\partial v_i'}{\partial x_j}}_{[2]} + v_i' v_j'\frac{\partial v_i'}{\partial x_j}\right) =$$

$$= -v_i'\frac{\partial P'}{\partial x_i} + \mu v_i'\frac{\partial^2 v_i'}{\partial {x_j}^2} + \rho v_i'\frac{\partial \overline{v_i' v_j'}}{\partial x_j}$$

Derivation of TKE equation

Term [1]: $v_i' \frac{\partial v_i'}{\partial t} = \frac{1}{2} \frac{\partial \left(v_i' v_i' \right)}{\partial t} \xrightarrow{\textit{time avg.}} \overline{\frac{1}{2} \frac{\partial \left(v_i' v_i' \right)}{\partial t}} = \frac{1}{2} \frac{\partial \overline{v_i' v_i'}}{\partial t} = \frac{\partial k}{\partial t}$

Term [2]: $v_i' \overline{v}_j \frac{\partial v_i'}{\partial x_j} = \overline{v}_j \frac{1}{2} \frac{\partial v_i' v_i'}{\partial x_j} \xrightarrow{\textit{time avg.}} \overline{\overline{v}_j \frac{1}{2} \frac{\partial v_i' v_i'}{\partial x_j}} = \overline{v}_j \frac{\partial k}{\partial x_j}$

Step 4: Take time average

$$\rho \left(\frac{\partial k}{\partial t} + \overline{v_i' v_j'} \frac{\partial \overline{v}_i}{\partial x_j} + \overline{v}_j \frac{\partial k}{\partial x_j} + \overline{v_i' v_j' \frac{\partial v_i'}{\partial x_j}} \right) =$$

$$= -\overline{v_i' \frac{\partial P'}{\partial x_i}} + \mu \, \overline{v_i' \frac{\partial^2 v_i'}{\partial {x_j}^2}} + \cancel{\rho \overline{v_i' \frac{\partial \overline{v_i' v_j'}}{\partial x_j}}}$$

Next, we rearrange some terms of this equation.

Derivation of TKE equation

First:

$$-\overline{v_i' \frac{\partial P'}{\partial x_i}} = -\frac{\partial \overline{P' v_i'}}{\partial x_i} + \overline{P' \frac{\partial v_i'}{\partial x_i}}$$

Second, since:

$$\frac{\partial^2 v_i' v_i'}{\partial {x_j}^2} = \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}\left(v_i' v_i'\right)\right] = \frac{\partial}{\partial x_j}\left(2 v_i' \frac{\partial v_i'}{\partial x_j}\right) = 2\frac{\partial v_i'}{\partial x_j} \cdot \frac{\partial v_i'}{\partial x_j} + 2 v_i' \frac{\partial^2 v_i'}{\partial {x_j}^2}$$

we can rewrite:

$$\overline{v_i' \frac{\partial^2 v_i'}{\partial {x_j}^2}} = \frac{1}{2}\frac{\partial^2 \overline{v_i' v_i'}}{\partial {x_j}^2} - \overline{\frac{\partial v_i'}{\partial x_j} \cdot \frac{\partial v_i'}{\partial x_j}}$$

Last:

$$\frac{\partial v_i' v_j' v_i'}{\partial x_j} = \underbrace{\cancel{v_i' v_i' \frac{\partial v_j'}{\partial x_j}}}_{=0 \text{ from Cont.}} + v_j' \frac{\partial v_i' v_i'}{\partial x_j} = 2 v_i' v_j \frac{\partial v_i'}{\partial x_j}$$

Derivation of TKE equation

Replacing into the equation yields:

$$\rho\underbrace{\left(\frac{\partial k}{\partial t}+\overline{v}_j\frac{\partial k}{\partial x_j}\right)}_{\mathrm{D}k/\mathrm{D}t}=-\rho\overline{v_i'v_j'}\frac{\partial\overline{v}_i}{\partial x_j}-\overline{\rho v_i'v_j'\frac{\partial v_i'}{\partial x_j}}-\frac{\partial\overline{P'v_i'}}{\partial x_i}+\overline{P'\frac{\partial v_i'}{\partial x_i}}+$$

$$+\frac{1}{2}\mu\frac{\partial^2\overline{v_i'v_i'}}{\partial {x_j}^2}-\mu\left(\overline{\frac{\partial v_i'}{\partial x_j}\cdot\frac{\partial v_i'}{\partial x_j}}\right)$$

This equation can be rewritten in a more compact form as:

$$\boxed{\frac{\mathrm{D}k}{\mathrm{D}t}=P_k-T_k-\Pi_k+\Phi_k+D_k-\mathcal{E}_k}$$

where:

- $P_k=-\rho\overline{v_i'v_j'}\frac{\partial\overline{v}_i}{\partial x_j}$ = **Production term** (production of TKE by the mean shear $\frac{\partial\overline{v}_i}{\partial x_j}$)

Derivation of TKE equation

- $T_k = \rho\overline{v_i' v_j' \frac{\partial v_i'}{\partial x_j}} = \frac{1}{2}\rho\frac{\partial \overline{v_i' v_j' v_i'}}{\partial x_j}$ = **Turbulent transport term**

(turbulent transport of TKE by the Reynolds stresses $-\rho\overline{v_i' v_j'}$)

- $\Pi_k = \frac{\partial \overline{P' v_i'}}{\partial x_i}$ = **Pressure diff. term** (transp. of TKE by press.)

- $\Phi_k = \overline{P' \frac{\partial v_i'}{\partial x_i}}$ = **Pressure strain diffusion term** (redistribution

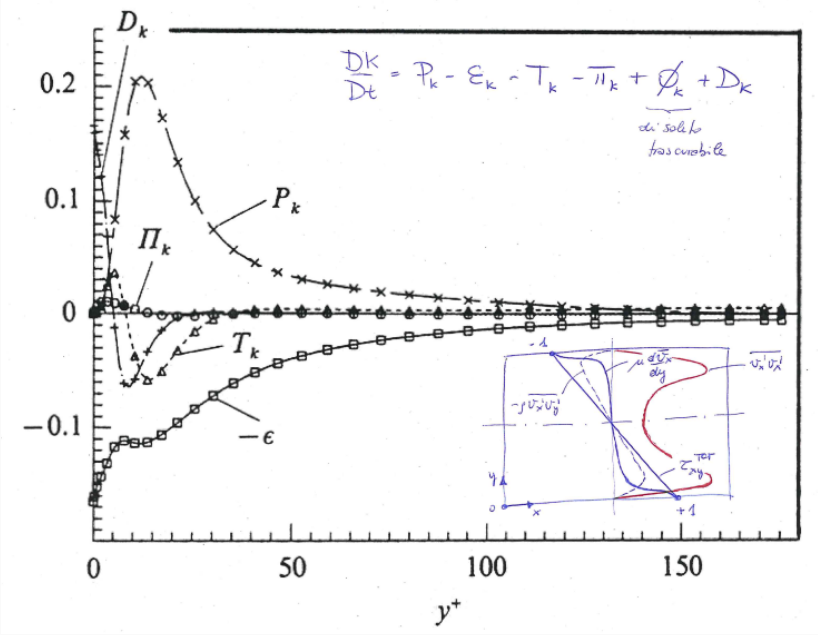
of energy due to pressure fluctuations)

- $D_k = \frac{1}{2}\mu\,\frac{\partial^2 \overline{v_i' v_i'}}{\partial {x_j}^2}$ = **Molecular viscous transport term**

(transport of TKE by viscous stresses)

- $\mathcal{E}_k = \mu\left(\overline{\frac{\partial v_i'}{\partial x_j} \cdot \frac{\partial v_i'}{\partial x_j}}\right)$ =**Dissipation term** (dissipation of TKE

due to fluctuations of viscous stresses)



D_k
0.2
0.1
0
−0.1
P_k
Π_k
T_k
$-\epsilon$
0
50
100
150
y^+
$\frac{DK}{Dt} = P_k - \varepsilon_k - T_k - \Pi_k + \phi_k + D_k$
di solito trascurabile

Reynolds' stress transport

The transport equation of TKE represents the starting point for nearly all turbulence models developed to improve Prandtl's mixing length model, which is the very first turbulence model ever proposed.

Before digging into the relation between TKE and mixing length, let us see another possible way of deriving the TKE transport equation.

Recalling that

$$\boxed{k = \frac{1}{2}\rho \cdot \mathrm{Tr}\left(\overline{v_i' \, v_j'}\right)}$$

we can obtain the TKE transport equation directly from the transport equation of the Reynolds' stresses.

Reynolds' stress transport

Without derivation, the transport equation of the Reynolds' stresses is:

$$\frac{\mathrm{D}\left(\overline{v_i' \, v_j'}\right)}{\mathrm{D}t} = P_{ij} + \Phi_{ij} - \Pi_{ij} - T_{ij} + D_{ij} - \mathcal{E}_{ij}$$

where:

P_{ij} = **Production term** (production of turbulent stress through interaction with mean strain rate $\frac{\partial \overline{v}}{\partial x}$):

$$P_{ij} = -\left(\overline{v_i' v_k'}\frac{\partial \overline{v}_j}{\partial x_k} + \overline{v_j' v_k'}\frac{\partial \overline{v}_i}{\partial x_k}\right)$$

Reynolds' stress transport

Φ_{ij} = **Pressure strain term** (redistribution of Reynolds stresses due to pressure fluctuations):

$$\Phi_{ij} = \overline{\frac{P'}{\rho}\left(\frac{\partial v_j'}{\partial x_i} + \frac{\partial v_i'}{\partial x_j}\right)}$$

Π_{ij} **Pressure transport term** (transport due to pressure fluctuations, usually negligible):

$$\Pi_{ij} = \frac{1}{\rho}\left(\frac{\partial \overline{p' v_j'}}{\partial x_i} + \frac{\partial \overline{p' v_i'}}{\partial x_j}\right)$$

Reynolds' stress transport

T_{ij} = **Turbulent pseudo-diffusion term** (pseudo-diffusion of Reynolds' stresses due to turbulent vel. fluctuations):

$$T_{ij} = \frac{\partial v_i' v_j' v_k'}{\partial x_k}$$

Note that $v_i' v_j' v_k'$ can be interpreted as the transport of $v_i' v_j'$ in the direction k, or the transport of $v_j' v_k'$ in i direction, and so on...

D_{ij} = **Molecular pseudo-diffusion term**:

$$D_{ij} = \nu \frac{\partial^2 \overline{v_i' v_j'}}{\partial {x_k}^2}$$

$\mathcal{E}_{ij}$ = **Dissipation term** (dissipation due to fluctuations of viscous stresses):

$$\mathcal{E}_{ij} = -2\nu \left(\overline{\frac{\partial v_i'}{\partial x_k} \frac{\partial v_j'}{\partial x_k}} \right)$$

Reynolds' stress transport

Taking the trace of the transport equation of the Reynolds' stresses and multiplying by 1/2 yields:

$$\frac{1}{\rho}\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{1}{2}\mathrm{Tr}[P_{ij}] + \frac{1}{2}\mathrm{Tr}[\Phi_{ij}] - \frac{1}{2}\frac{\partial C_{iij}}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\nu\frac{\partial\left(\overline{v_i'v_i'}\right)}{\partial x_j}\right] - \frac{1}{2}\mathrm{Tr}[\mathcal{E}_{ij}]$$

where:

$$C_{iij} = \overline{v_i'v_i'v_j'} + \frac{1}{\rho}\overline{P'v_i'}\delta_{ij} + \frac{1}{\rho}\overline{P'v_j'}\delta_{ij}$$

since:

$$C_{ijk} = \overline{v_i'v_j'v_k'} + \frac{1}{\rho}\overline{P'v_i'}\delta_{jk} + \frac{1}{\rho}\overline{P'v_j'}\delta_{ik}$$

and:

$$\frac{1}{2}\mathrm{Tr}[\Phi_{ij}] = \Phi_{ii} = \frac{1}{2}\overline{\frac{P'}{\rho}\left(\frac{\partial v_i'}{\partial x_i} + \frac{\partial v_i'}{\partial x_i}\right)} = \overline{\frac{P'}{\rho}\frac{\partial v_i'}{\partial x_i}}$$

Plugging in these expression yields the TKE transport equation.

$k - \mathcal{E}$ model

TKE equation

To understand where the $k - \mathcal{E}$ model comes from, let us consider the eddy viscosity:

$$\mu^e = \rho \ell_{mix}^2 \left| \frac{\partial \overline{v}_x}{\partial y} \right|$$

and ℓ_{mix} is the mixing length.

We can correlate directly k and $\nu^e = \mu^e / \rho$ via dimensional analysis:

$$\left. \begin{aligned} [\nu^e] &= \left[m^2/s\right] \\ [k] &= \left[m^2/s^2\right] \end{aligned} \right\} \qquad \nu^e \propto k^{1/2}\,\ell$$

and

$$\boxed{\nu^e = C_\mu\, k^{1/2}\,\ell}$$

where C_μ = proportionality constant and ℓ = characteristic length of the flow, not necessarily equal to ℓ_{mix}.

Characteristic length

We need an expression for ℓ. One possiblilty is to use:

$$\left.\begin{aligned}[k] &= \left[m^2/s^2\right] \\ [\mathcal{E}] &= \left[m^2/s^3\right]\end{aligned}\right\} \qquad \ell = \frac{k^{3/2}}{\mathcal{E}}$$

and obtain:

$$\nu^e = C_\mu k^{1/2}\left(\frac{k^{3/2}}{\mathcal{E}}\right) \rightarrow \boxed{\nu^e = C_\mu \frac{k^2}{\mathcal{E}}}$$

Note that the characteristic velocity and time of the flow, corresponding to ℓ will be:

$$\tau = \frac{k}{\mathcal{E}}\ [s]$$

$$u = \frac{\ell}{\tau} = k^{1/2}\ [m/s]$$

Note: from now on, we will use the more common ν^t instead of ν^e.

$k - \mathcal{E}$ model

The $k - \mathcal{E}$ model represents the eddy viscosity as:

$$\boxed{\nu^t = C_\mu \frac{k^2}{\mathcal{E}}}$$

and needs therefore two equations: One for k and one for $\mathcal{E}$ (+ 3 RANS equations + Continuity equation) to close the model. For this reason, it belongs to the class of **Two-Equation Turbulence Models**.

The transport equation for k is the one we just derived, written as:

$$\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = P_k - \mathcal{E} - \frac{\partial T'}{\partial x_j}$$

where Φ_k has been neglected and all terms representing transport by some diffusion mechanisms have been included in a single term:

$k - \mathcal{E}$ model

$$T' := \frac{1}{2}\rho\overline{v_i'v_i'v_j'} + \overline{P'v_j'} - \frac{1}{2}\mu\frac{\partial\overline{v_i'v_i'}}{\partial x_j}$$

such that $\dfrac{\partial T'}{\partial x_j} = T_k + \Pi_k - D_k$.

T' is the term to be modelled in order to *close* the TKE transport equation. To this aim, we use the **gradient diffusion model** to write:

$$\boxed{T' = -\frac{\nu^t}{\sigma_k}\frac{\partial k}{\partial x_j}} \quad [*]$$

where the closure coefficient σ_k is the analogous of the Prandtl number for the transport of TKE:

$$\sigma_k = \frac{\text{Diffusivity of the momentum}}{\text{Diffusivity of the TKE via turbulent transport}}$$

$k - \mathcal{E}$ model

Based on [∗], the **transport equation for the TKE** reads as:

$$\boxed{\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = P_k - \mathcal{E} + \frac{\partial}{\partial x_j}\left[\frac{\nu^t}{\sigma_k}\frac{\partial k}{\partial x_j}\right]}$$

while the **transport equation for** $\mathcal{E}$ (derivation is omitted) is:

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \overline{u}_j \frac{\partial \mathcal{E}}{\partial x_j} = C_{\mathcal{E}1} P_k \frac{\mathcal{E}}{k} - C_{\mathcal{E}2}\frac{\mathcal{E}^2}{k} + \frac{\partial}{\partial x_j}\left[\frac{\nu^t}{\sigma_{\mathcal{E}}}\frac{\partial \mathcal{E}}{\partial x_j}\right]}$$

where the closure coefficient

$$\sigma_{\mathcal{E}} = \frac{\text{momentum diffusivity}}{\text{diffusivity of turbulent dissipation via turbulent transport}}$$

is the analogous of the Prandtl number for the transport of the turbulent dissipation rate.

$k - \mathcal{E}$ model

These two equations are coupled with the Continuity and RANS equations:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_j} + \left(\frac{\mu + \mu^t}{\rho}\right)\frac{\partial^2 \overline{u}_i}{\partial {x_j}^2}$$

where

$$\frac{\mu + \mu^t}{\rho} = \nu + \nu^t$$

and $\nu^t = C_\mu \frac{k^2}{\mathcal{E}}$. Their solution is needed in order to solve the two equations of the model.

$k - \mathcal{E}$ model

Values of the constants. For the *Standard $k - \mathcal{E}$ model*:

1) $\boxed{C_\mu = 0.09}$

This value comes from observations that, in *turbulent shear flow*:

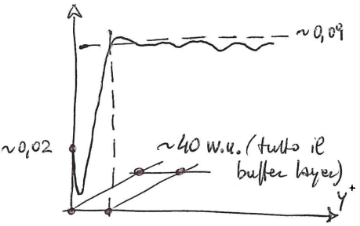
$$\nu^t = \frac{\left|\overline{v_i' v_j'}\right|}{P_k} \quad \Rightarrow \quad \frac{\left|\overline{v_i' v_j'}\right|}{k} = \sqrt{C_\mu \frac{P_k}{\mathcal{E}}} \quad \Rightarrow \quad C_\mu = \left(\frac{\left|\overline{u_i' u_j'}\right|}{k}\right)^2 \cdot \frac{\mathcal{E}}{P_k}$$

and that $\dfrac{\left|\overline{v_i' v_j'}\right|}{k} \simeq 0.3$ if $P_k \sim \mathcal{E}$;
which yields $C_\mu = 0.3^2 \cdot 1 = 0.09$.

However, in channel flow

$$C_\mu = \nu^t \mathcal{E} / k^2$$

is not uniform in the wall-normal dir.



$k - \mathcal{E}$ model

2) $\boxed{\sigma_k = 1.0}$

This value implies that momentum diffusivity is equal to TKE diffusivity: If this is not the case, than the model might not be reliable anymore.

3) $\boxed{\sigma_{\mathcal{E}} = 1.3}$

4) $\boxed{C_{\mathcal{E}1} = 1.44}$

5) $\boxed{C_{\mathcal{E}2} = 1.92}$

This value is obtained fitting experimental data obtained for *grid turbulence*: Grid turbulence is homogeneous and characterized by a spatially-decaying intensity along the mean flow direction, as the flow moves away from the grid. This flow features are not observed in pipe/channel flow, for instance...

$k - \mathcal{E}$ model

Other sets of values are available in the literature. For example:

$$C_\mu = 0.0845, \;\; \sigma_k = \sigma_{\mathcal{E}} = 0.72, \;\; C_{\mathcal{E}1} = 1.42, \;\; C_{\mathcal{E}2} = 1.68$$

When this set of values is used, we refer to the *$k - \mathcal{E}$ RNG (Re-Normalisation Group) model*.

$$C_\mu = \frac{1}{A_0 + A_s \frac{k}{\mathcal{E}}}, \;\; \sigma_k = 1.0, \;\; \sigma_{\mathcal{E}} = 1.2, \;\; C_{\mathcal{E}1} = 1.44, \;\; C_{\mathcal{E}2} = 1.9$$

with $A_0 = 4.04$ and $A_s = f(\frac{\partial v_i}{\partial x_j})$. When this set of values is used, we refer to the *Realisable $k - \mathcal{E}$ model*.

Note on Gradient Diffusion model

Consider a generic scalar function Φ. Then:

- Flux of Φ: $\vec{v}\Phi$
- Turbulent flux of Φ: $\vec{v'}\Phi'$

since $\vec{v} = \langle\vec{v}\rangle + \vec{v'}$ and $\Phi = \langle\Phi\rangle + \Phi'$

- Mean gradient of Φ: $-\overline{\nabla}\langle\Phi\rangle$

Gradient diffusion hypothesis: The mean turbulent flux of Φ occurs in the direction of the mean gradient of Φ and is proportional to it:

$$\langle\vec{v'}\Phi'\rangle \propto -\overline{\nabla}\langle\Phi\rangle$$

Therefore, we can define a positive scalar $\Gamma^t(\vec{x}, t)$ such that:

$$\boxed{\langle\vec{v'}\Phi'\rangle = -\Gamma^t \cdot \overline{\nabla}\langle\Phi\rangle} \quad [\star]$$

Eq. $[\star]$ is analogous to Fourier's law and to Fick's law.

Note on Gradient Diffusion model

Physical meaning of $\Gamma^t(\vec{x}, t)$: turbulent diffusivity (=turbulent diffusion coefficient).

Analogy with:

$$-\langle v_i' v_j' \rangle = -\nu^t \cdot \overline{\overline{\nabla}} \langle \vec{v} \rangle$$

Indeed, consider the transport equation for Φ:

$$\frac{\partial \Phi}{\partial t} + \overline{\overline{\nabla}}(\vec{v} \cdot \Phi) = \Gamma \; \overline{\overline{\nabla}}^2 \Phi$$

Take average:

$$\frac{\partial \langle \Phi \rangle}{\partial t} + \overline{\overline{\nabla}}(\langle \vec{v} \cdot \Phi \rangle) = \Gamma \; \overline{\overline{\nabla}}^2 \langle \Phi \rangle \qquad [1]$$

with:

$$\langle \vec{v} \cdot \Phi \rangle = \langle \vec{v} \rangle \cdot \langle \Phi \rangle + \underbrace{\langle \vec{v'} \cdot \Phi' \rangle}_{\text{Turb. flux of } \Phi} \qquad [2]$$

Note on Gradient Diffusion model

Plug [2] into [1] to get:

$$\underbrace{\frac{\partial\langle\Phi\rangle}{\partial t} + \overline{\nabla}(\langle\vec{v}\rangle\cdot\langle\Phi\rangle)}_{\frac{\mathrm{D}\langle\Phi\rangle}{\mathrm{D}t}} + \overline{\nabla}(\langle\vec{v'}\cdot\Phi'\rangle) = \Gamma\ \overline{\nabla}^2\langle\Phi\rangle$$

$$\frac{\mathrm{D}\langle\Phi\rangle}{\mathrm{D}t} = \Gamma\ \overline{\nabla}^2\langle\Phi\rangle - \overline{\nabla}(\langle\vec{v'}\cdot\Phi'\rangle)$$

Using [$\star$]:

$$\frac{\mathrm{D}\langle\Phi\rangle}{\mathrm{D}t} = \Gamma\ \overline{\nabla}^2\langle\Phi\rangle - \overline{\nabla}(\Gamma^t\ \overline{\nabla}\langle\Phi\rangle)$$

Analogy with RANS:

$$\frac{\mathrm{D}\langle\vec{v}\rangle}{\mathrm{D}t} = \nu\ \overline{\nabla}^2\langle\vec{v}\rangle - \overline{\nabla}(\nu^t\ \overline{\nabla}\langle\vec{v}\rangle)$$

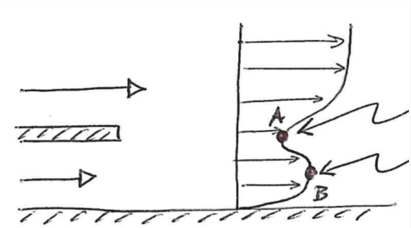
Weaknesses of the $k - \mathcal{E}$ model

1) Gradient diffusion assumption

Based on this assumption, we can set $\mu^t = \rho\Gamma^t$ and, in turn:

$$\tau_{xy}^t = -\rho\overline{v_x' v_y'} = \mu^t \frac{\partial \overline{v}_x}{\partial y}$$

such that $\tau_{xy}^t = 0$ if $\frac{\partial \overline{v}_x}{\partial y} = 0$. However, this is not always true:



In A and B, $\frac{\partial \overline{v}_x}{\partial y} = 0$ (local vel. minimum), but τ_{xy}^t might be different from 0. Yet the model would yield $\tau_{xy}^t = 0$!

2) Isotropic eddy viscosity

The model assumes isotropic eddy viscosity ν^t: This is not true in flows that are strongly 3D, or have significant curvature effects.

Weaknesses of the $k - \mathcal{E}$ model

3) Overestimation of k

In the presence of strong deformation in the direction normal to the mean flow (e.g. near the wall in channel flow), k is overestimated, and so is $\nu^t \propto k$.

Too high values of k and ν^t lead to inaccurate prediction of the flow structure and of flow separation phenomena.

4) Underestimation of $\mathcal{E}$

In flows with separation, $\mathcal{E}$ is underestimated near the wall, so the energy of the flow is overestimated. This can lead to a "delay" of flow separation, but also to ovestimation of the heat transfer rates.

Despite these disadvantages, the $k - \mathcal{E}$ model is still one of the most popular in RANS-based CFD codes.

Alternative turbulence models

$k-\omega$ model

The $k-\omega$ model is a Two-Equation model alternative to $k-\mathcal{E}$. The kinetic energy is kept as model variable, while dissipation is replaced by vorticity:

$$\omega = \frac{\mathcal{E}}{k}$$

Putting $\mathcal{E} = \omega \cdot k$ in the transport equation for $\mathcal{E}$, one finds:

$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = \frac{\partial}{\partial x_j}\left(\frac{\nu^t}{\sigma_\omega}\frac{\partial \omega_i}{\partial x_j}\right) + (C_{\mathcal{E}1} - 1)\frac{P_k \omega_i}{k} - (C_{\mathcal{E}2} - 1)\,\omega_i^2 + \frac{2\nu^t}{\sigma_\omega k}\frac{\partial \omega_i}{\partial x_j}\frac{\partial k}{\partial x_j}$$

The $k-\omega$ model:

1) is a low Reynolds number model,

2) works better than $k-\mathcal{E}$ with wall-bounded flows, high-curvature flows, flows with separation, jets,

3) has lower convergence rate and higher sensitivity to initial conditions wrt $k-\mathcal{E}$.

Algebraic models

These models are also called **Zero-Equation Models**.

1) Mixing Length model (Prandtl)

$$\tau_{ij}^t = -\rho\overline{v_i' v_j'} \;\Rightarrow\; \frac{\tau_{ij}^t}{\rho} \simeq \nu^t \frac{\partial \overline{v}_i}{\partial x_j}$$

Recall: Dimensional analysis shows that

$$[\nu^t] = \left[\frac{m^2}{s}\right] \sim \ell \Delta v$$

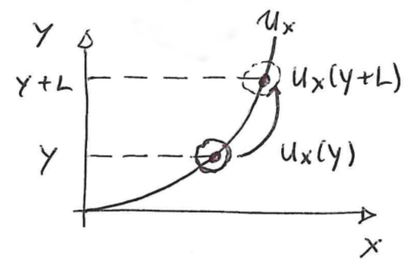
where ℓ is the mixing length (length above which flow structures lose their coherence, momentum and energy).

Algebraic models

Velocity fluctuation over a distance ℓ:

$$\Delta v \sim \ell \left| \frac{\partial \overline{v}_i}{\partial x_j} \right|$$

$$\nu^t = \ell^2 \left| \frac{\partial \overline{v}_i}{\partial x_j} \right|$$



$$\frac{\partial \overline{v}_x}{\partial y} \simeq \frac{\overline{v}_x(y+l) - v_x(y)}{L} = \frac{\Delta v}{\ell}$$

Mixing length model for τ^t:

$$\frac{\tau^t_{ij}}{\rho} = \ell^2 \left| \frac{\partial \overline{v}_i}{\partial x_j} \right| \left(\frac{\partial \overline{v}_i}{\partial x_j} \right)$$

$$\Rightarrow \; \Delta v = \ell \left(\frac{\partial \overline{v}_x}{\partial y} \right)$$

The model is simple and of practical use for engineering calculations provided that an estimate for ℓ is available.

Algebraic models

Notes on the mixing length model:
1) The model is based on the *gradient diffusion* hypothesis too, so it predicts zero flux (no transport) anytime the mean vel. gradient is zero: This is not always true, especially in complex flows (e.g. wall-bounded or with separation).

2) The *mixing length is not universal*: It is flow-depending and may even change in different locations within the same flow.

Algebraic models

2) Smagorinsky model

$$\boxed{\nu^t = 2\ell^2 \left(\overline{e}_{ij} \ \cdot \overline{e}_{ij}\right)^{1/2}}$$

where $\overline{e}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{v}_j}{\partial x_i} + \frac{\partial \overline{v}_i}{\partial x_j}\right)$ = mean strain rate.

3) Baldwin and Lomax model

$$\boxed{\nu^t = \ell^2 \left(2\overline{\Omega}_{ij}\overline{\Omega}_{ij}\right)^{1/2}}$$

where $\overline{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{v}_j}{\partial x_i} - \frac{\partial \overline{v}_i}{\partial x_j}\right)$ = mean rotation rate.

Note: Any tensor J can be decomposed in a symmetric part and an antisymmetric part: $S = (J + J^T)/2$, $A = (J - J^T)/2$, respectively. In this case, $J = \partial \overline{v}_i / \partial x_j$, $S = e$ and $A = \Omega$.

One-Equation models

One-Equation models require just one transport equation for one turbulent quantity: The TKE. Such transport equation is necessary in order to evaluate the eddy viscosity, according to the following expression:

$$\nu^t \sim \ell \Delta v \sim \ell k^{1/2}$$

$$\boxed{\nu^t = \mathcal{C} \ell k^{1/2}} \quad \circledast$$

where $\mathcal{C}$ is a constant.

To compute ν^t, we must:

1) specify the mixing length $\ell = \ell(\vec{x}, t)$

One-Equation models

2) determine $k = k(\vec{x}, t)$ from the transport equation

$$\frac{\mathrm{D}k}{\mathrm{D}t} = \overline{\nabla} \cdot \left(\frac{\nu^t}{\sigma_k} \overline{\nabla} k \right) + P_k - \mathcal{E} \quad (*)$$

3) model $\mathcal{E}$ as

$$\mathcal{E} = \mathcal{C}_D \frac{k^{3/2}}{\ell} \quad (*)$$

with constant $\mathcal{C}_D$. Since $\ell = \dfrac{\nu^t}{\mathcal{C} k^{1/2}}$, we obtain

$$\mathcal{E} = \mathcal{C} \cdot \mathcal{C}_D \frac{k^2}{\nu^t} \;\Rightarrow\; \frac{\nu^t \mathcal{E}}{k^2} = \mathcal{C} \cdot \mathcal{C}_D = \mathrm{const.}$$

One-equation models

In addition to eqns. marked with (*), the **turbulent viscosity hypothesis** is also imposed:

$$\boxed{\overline{v_i' v_j'} = \frac{2}{3} k \delta_{ij} - \nu^t \left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i} \right)}$$

This hypothesis assumes that the deviatoric part of the Reynolds' stress $-\rho \overline{v_i' v_j'} + \frac{2}{3} \rho k \delta_{ij}$ is proportional to the mean strain rate $\rho \nu^t (\partial \overline{v}_i / \partial x_j + \partial \overline{v}_j / \partial x_i)$.