Modelling of Turbulent Flows

Lecture 4.2

April 27, 2020

Reynolds Stress Modelling and the Law of the Wall

Reynolds Stress Modelling

Following the Reynolds procedure, we determined the following form for the averaged Navier-Stokes equations:

$$\rho \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho \overline{v_j' v_j'}\right]$$

with ρ and μ uniform.

We examined the physical meaning of the Reynolds stresses and we concluded that they are always (probably!) negative so that they contribute with an extra drag.

The analysis of a fluid parcel we made is much like the analysis of the molecular motion done by Boussinesq (1877), suggesting a possible way to model the Reynolds stresses with a “model” viscosity:

$$\tau_{yx}^{(t)} = -\rho \overline{v_x' v_y'} = \mu^e \left[\frac{\partial \bar{v}_x}{\partial y} + \frac{\partial \bar{v}_y}{\partial x} \right]$$

in which the superscript “e” indicates “eddy” and μ^e is not a fluid property, but rather it represents the action of turbulence on fluid motion.

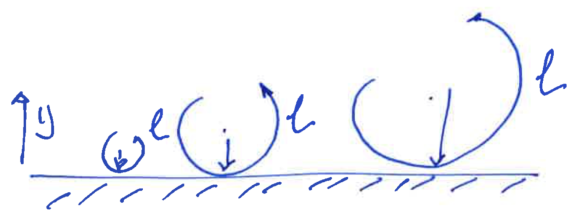
Observation: at the wall $\overline{v_i' v_j'} = 0$, and therefore $\mu^e = 0$.

This implies that $\mu^e = \mu^e(y)$ where y is the wall distance in the reference case of channel/pipe flow.

Prandtl proposed the following equation/model for the eddy viscosity:

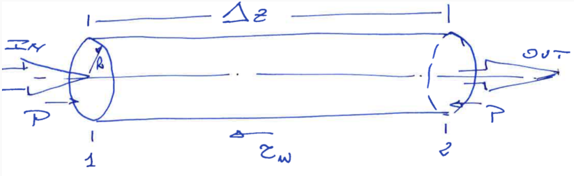
$$\mu^e = \rho l^2 \left| \frac{\mathrm{d}\bar{v}_x}{\mathrm{d}y} \right|$$

where l is the *mixing length* with the same physical meaning of the "mean free path" of molecules in the kinetic theory of gases.
In addition $l = l(y)$.
Prandtl hypothesized $l \propto y$ following the idea that the farther from the wall, the larger the radius of the vortex which mixes the flow.



Turbulent Flow in a Pipe

Let us consider the flow driven by a pressure gradient in a pipe (Poiseuille flow), characterized by a turbulent Reynolds number:



The force balance gives:

$$P_1 - P_2 = \Delta P = \frac{\mathrm{d}P}{\mathrm{d}z}\Delta z$$

$$\pi R^2 P_2 - \pi R^2 P_1 = \Delta z \cdot 2\pi R \tau_w$$

$$R[P_2 - P_1] = \Delta z \cdot 2\tau_w$$

$$\boxed{\frac{\Delta P}{\Delta z} = -\frac{2}{R}\tau_w} \quad \text{Ⓐ}$$

In cylindrical coordinates we have

$$\bar{v}_z = \bar{v}_z(r) \;\; ; \quad \bar{v}_r = \bar{v}_\theta = 0 \;\; ; \quad \frac{\mathrm{d}\bar{P}}{\mathrm{d}z} = \mathrm{const.}$$

The only relevant component of N-S is the z component:

$$\rho\left[\frac{\partial\bar{v}_z}{\partial t}+\bar{v}_r\frac{\partial\bar{v}_z}{\partial r}+\frac{\bar{v}_\theta}{r}\frac{\partial\bar{v}_z}{\partial\theta}+\bar{v}_z\frac{\partial\bar{v}_z}{\partial z}\right]=$$

$$-\frac{\mathrm{d}\bar{P}}{\mathrm{d}z}+\frac{1}{r}\frac{\partial}{\partial r}(r\bar{\tau}_{rz}+r\bar{\tau}^e_{rz})+$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\cancel{\bar{\tau}_{\theta z}}+\cancel{\bar{\tau}^e_{\theta z}})+\frac{\partial}{\partial z}(\cancel{\bar{\tau}_{zz}}+\cancel{\bar{\tau}^e_{zz}})$$

and finally

$$-\frac{\mathrm{d}\bar{P}}{\mathrm{d}z}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r(\bar{\tau}_{rz}+\bar{\tau}^e_{rz})=0$$

NOTE: it is exactly the same as in the laminar case, but with the presence of $\bar{\tau}^e_{rz}$. Upon integration:

$$\bar{\tau}_{rz}+\bar{\tau}^e_{rz}=\frac{r}{2}\frac{\mathrm{d}\bar{P}}{\mathrm{d}z}$$

Equation Ⓐ is:

$$\frac{\Delta P}{\Delta z} = -\frac{2}{R}\tau_w \quad \Rightarrow \quad \frac{\mathrm{d}\bar{P}}{\mathrm{d}z} = -\frac{2}{R}\bar{\tau}_w$$

which, substituted into the previous one, yields:

$$\boxed{\bar{\tau}_{rz} + \bar{\tau}^e_{rz} = -\frac{r}{R}\bar{\tau}_w} \quad \text{Ⓑ}$$

We know that:

$$\bar{\tau}_{rz} = \mu\frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r}\ ; \quad \bar{\tau}^e_{rz} = -\rho\overline{v'_z v'_r}$$

and adopting the Prandtl mixing lenght model, we have:

$$\bar{\tau}^e_{rz} = -\rho\overline{v'_z v'_r} = \underbrace{\overline{\rho l^2 \left|\frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r}\right| \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r}}}_{\text{Ⓒ}} = \mu^e\frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r}$$

In pipe flow, turbulence is generated at the wall and we want to compute the velocity profile as a function of the wall distance, not of the radius. We apply the following coordinate change:

$$y = R - r \quad \Rightarrow \quad \mathrm{d}y = -\mathrm{d}r$$

Then equation Ⓑ plus Ⓒ becomes:

$$\mu \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r} + \mu^e \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r} = -\frac{r}{R}\bar{\tau}_w$$

$$\mu \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r} + \rho l^2 \left| \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r} \right| \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r} + \frac{r}{R}\bar{\tau}_w = 0$$

and with the coordinate change:

$$-\mu \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}y} - \rho l^2 \left| \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}y} \right| \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}y} + \frac{R-y}{R}\bar{\tau}_w = 0$$

Note on the absolute value

In the Prandtl's mixing length model for the viscosity, the absolute value is necessary to avoid possible values of negative viscosity. However, in our case the derivative of the velocity profile (average) is always positive and we can safely remove the absolute value.

$$\boxed{\rho l^2 \left(\frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r}\right)^2 + \mu \frac{\mathrm{d}\bar{v}_z}{\mathrm{d}r} - \left(1 - \frac{y}{R}\right)\bar{\tau}_w = 0}$$

which is a *first-order differential equation, non-linear in* $\bar{v}_z$.
Boundary conditions are:

$$\begin{cases} \bar{v}_z = 0 & \text{at } y = 0 \\ \bar{v}_z = \bar{v}_{z,max} & \text{at } y = R \end{cases}$$

Wall Units

As usual, we must now make the equation dimensionless and we need to identify proper scaling variables. We define the

$$\text{Shear velocity} \quad u^\star := \sqrt{\frac{\bar{\bar{\tau}}_w}{\rho}}$$

This velocity is characteristic of all the processes occurring at the wall and is useful to define the so-called **wall variables**:

$$\text{velocity} : u^\star := \sqrt{\frac{\bar{\bar{\tau}}_w}{\rho}} \quad \Rightarrow \quad u^+ := \frac{\bar{v}_z}{u^\star}$$
$$\text{length} : l^\star := \frac{\nu}{u^\star} \quad \Rightarrow \quad y^+ := \frac{y}{\nu/u^\star}$$

Our equation becomes:

$$\rho u^{\star 2} l^{+2} \left(\frac{\mathrm{d}u^+}{\mathrm{d}y^+}\right)^2 + \mu \frac{u^\star}{l^\star} \frac{\mathrm{d}u^+}{\mathrm{d}y^+} - \left(1 - \frac{y^+}{R^+}\right) \bar{\tau}_w = 0$$

$$\rho u^{\star 2} l^{+2} \left(\frac{\mathrm{d}u^+}{\mathrm{d}y^+}\right)^2 + \rho \frac{\nu}{l^\star} u^\star \frac{\mathrm{d}u^+}{\mathrm{d}y^+} - \left(1 - \frac{y^+}{R^+}\right) \bar{\tau}_w = 0$$

Remembering that $\bar{\tau}_w = \rho u^{\star 2}$ we have

$$l^{+2} \left(\frac{\mathrm{d}u^+}{\mathrm{d}y^+}\right)^2 + \frac{\mathrm{d}u^+}{\mathrm{d}y^+} - \left(1 - \frac{y^+}{R^+}\right) = 0$$

This equation is a quadratic form in $\frac{\mathrm{d}u^+}{\mathrm{d}y^+}$ and the solution is

$$\frac{\mathrm{d}u^+}{\mathrm{d}y^+} = \frac{-1 + \sqrt{1 + 4 l^{\star 2}(1 - y^+/R^+)}}{2 l^+}$$

which upon integration gives:

$$u^+ = \int_0^{y^+} \frac{-1 + \sqrt{1 + 4 l^{\star 2}(1 - y^+/R^+)}}{2 l^+} \mathrm{d}y^+ \quad \textcircled{D}$$

But we still have to define $l^\star$. According to **Prandtl**'s mixing length model:

$$l^+ = k\, y^+ \text{ with } \; k = 0.4$$

while the **Van Driest** relation is empirically based and it recovers the Prandtl's equation for $y^+ \gg A$:

$$l^+ = k\, y^+ \left[1 - e^{-y^+/A}\right] \text{ with } \; k = 0.4 \; ; \; A = 36$$

In equation Ⓓ there is no reminiscence of the scale of the system except for R^+ and if $y^+ \ll R^+$ the term y^+/R^+ becomes negligible compared with 1. This has a physical meaning in the sense that turbulence is produced and controlled by what happens near the wall. Mathematically, this can simplify the equation.

Universal Profile

If we neglect the term y^+/R^+ we obtain

$$u^+ = \int_0^{y^+} \frac{-1 + \sqrt{1 + 4l^{+2}}}{2l^{+2}} \mathrm{d}y^+$$

This equation depends only on the wall distance and therefore it can be *generalized for any type of turbulent flow over a wall* (in a pipe or in a channel). The solution of this equation will produce a **universal profile** which means that measurements obtained for different geometries and for different Reynolds numbers must overlap if plotted in terms of y^+ and u^+.

Let's look for the solution of this equation in two different regions: **inner flow** (near the wall) and **outer flow** (far from the wall).

Viscous Sublayer

Let us examine again the complete equation:

$$l^{+2} \underbrace{\left(\frac{\mathrm{d}u^+}{\mathrm{d}y^+}\right)^2}_{\text{Reynolds stress}} + \underbrace{\frac{\mathrm{d}u^+}{\mathrm{d}y^+}}_{\text{shear stress}} - \underbrace{\left(1 - \frac{y^+}{R^+}\right)}_{\text{pressure gradient}} = 0 \quad \text{Ⓔ}$$

The shear stress dominates near the wall, while the Reynolds stresses become important farther away from the wall.
To neglect the Reynolds stresses, it must be $l^+ \ll 1$ so that $l^{+2} \ll 1$. If we apply the Van Driest model we obtain:

$$y^+ = 10 \;\rightarrow\; l^+ = 1 \;\rightarrow\; l^{+2} = 1$$

$$y^+ = 5 \;\rightarrow\; l^+ = 0.259 \;\rightarrow\; l^{+2} = 0.067$$

so when we are very close to the wall the flow is viscosity-dominated (shear stress) and we call this region **viscous sublayer**.

Solution for the Viscous Sublayer (Inner Region)

$y^+ \ll R^+$ and equation Ⓔ becomes

$$\frac{\mathrm{d}u^+}{\mathrm{d}y^+} = 1 \;\Rightarrow\; u^+ = \int_0^{y^+} \mathrm{d}y^+$$

$\boxed{u^+ = y^+}$ which is reminiscent of *Couette flow*.

If we start from equation Ⓓ and we apply $y^+ \ll R^+$,

$$u^+ = \int_0^{y^+} \frac{-1+\sqrt{1+4l^{+2}}}{2l^{+2}} \mathrm{d}y^+ =$$

$$= \int_0^{y^+} \frac{2}{1+\sqrt{1+4l^{+2}}} \mathrm{d}y^+ \overset{l^+ \ll 1}{\simeq} \int_0^{y^+} \mathrm{d}y^+ = y^+$$

Solution for the Inertial Layer (Outer Region)

(Also called "Turbulent Core") Now we have to integrate from y^+ to R^+:

$$u^+\Big|_{y^+}^{R^+} = \int_{y^+}^{R^+} \frac{-1 + \sqrt{1 + 4l^{+2}\left(1 - \frac{y^+}{R^+}\right)}}{2l^{+2}} \mathrm{d}y^+$$

$$u^+ = u^+_{\mathrm{max}} + \int_{y^+}^{R^+} \frac{1 - \sqrt{1 + 4l^{+2}\left(1 - \frac{y^+}{R^+}\right)}}{2l^{+2}} \mathrm{d}y^+$$

For $l^+ \gg 1$ we have $1 - \sqrt{1 + 4l^{+2}\left(1 - \frac{y^+}{R^+}\right)} \simeq -2l^+\sqrt{1 - \frac{y^+}{R^+}}$ and

$$u^+ = u^+_{\mathsf{max}} - \int_{y^+}^{R^+} \frac{\sqrt{1 - \frac{y^+}{R^+}}}{l^+} \mathrm{d}y^+$$

Of course, the same equation can be derived also from equation (E) when we neglect the shear stress term:

$$l^{+2}\left(\frac{\mathrm{d}u^+}{\mathrm{d}y^+}\right)^2 + \cancel{\frac{\mathrm{d}u^+}{\mathrm{d}y^+}} - \left(1 - \frac{y^+}{R^+}\right) = 0$$

In the Outer Region we can apply Prandtl's model: $l^+ = ky^+$ and solve the integral

$$u^+ = u^+_{\text{max}} - \int_{y^+}^{R^+} \frac{\sqrt{1 - \frac{y^+}{R^+}}}{l^+} \mathrm{d}y^+ =$$

$$= u^+_{\text{max}} - \frac{1}{k} \int_{y^+}^{R^+} \frac{\sqrt{1 - \frac{y^+}{R^+}}}{\frac{y^+}{R^+}} \frac{\mathrm{d}y^+}{R^+}$$

So we want to solve $\int \frac{\sqrt{1-x}}{x}\mathrm{d}x$. If we do that (a little tedious), we obtain

$$u^+ = u^+_{\text{max}} + \frac{2}{k}\sqrt{1 - \frac{y^+}{R^+}} + \frac{1}{k}\ln\frac{1 - \sqrt{1 - \frac{y^+}{R^+}}}{1 + \sqrt{1 + \frac{y^+}{R^+}}}$$

We remark here that this solution is only valid far from the wall: Indeed, the logarithm diverges for $y^+ \to 0$!

Now, if $\frac{y^+}{R^+}$ is small, as it is when y^+ approaches R^+, then the square root can be approximated as follows:

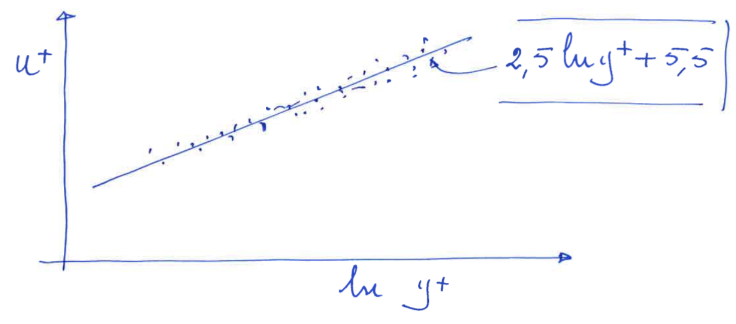
$$\sqrt{1 - \frac{y^+}{R^+}} \simeq 1 - \frac{1}{2}\frac{y^+}{R^+} + \dots$$

and then we have

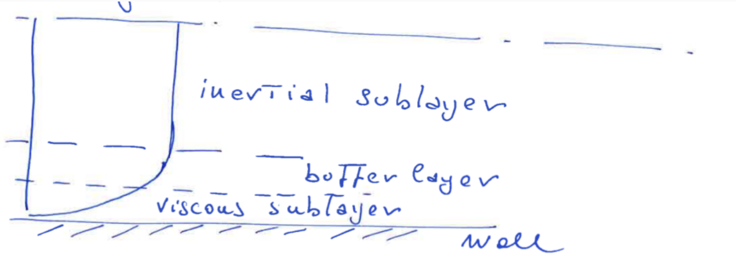
$$\begin{aligned} u^+ &= \frac{1}{k}\ln y^+ + \underbrace{u^+_{\text{max}}}_{\text{const.}} + \underbrace{\frac{2}{k}}_{\text{const.}} - \frac{1}{k}\frac{y^+}{R^+} \\ &= \frac{1}{k}\ln y^+ + [\text{const.}] + \mathcal{O}\left[\frac{y^+}{R^+}\right] \end{aligned}$$

Experimental Results

From experimental measurements, we have



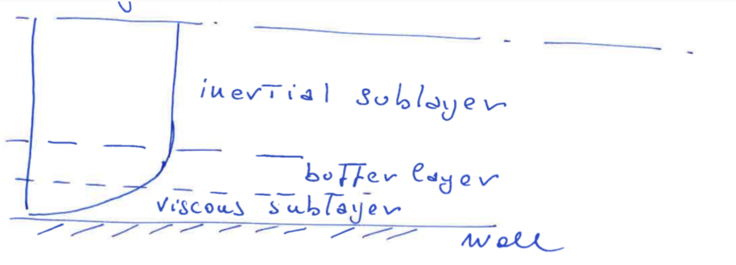
We have solved the equation in the following way:



1) Viscous sublayer: only the viscous term is important

$$u^+ = y^+$$

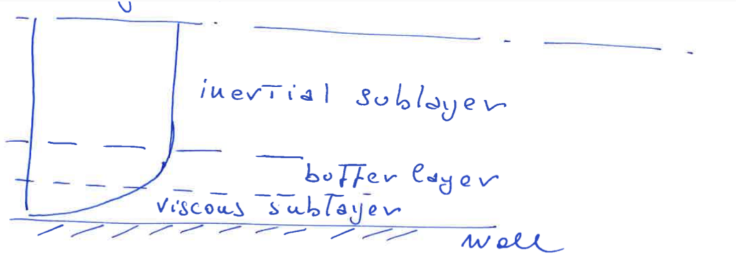
We have solved the equation in the following way:



2) Buffer sublayer: both viscous and inertial terms of the stresses are important

$$u^{+} = ?$$

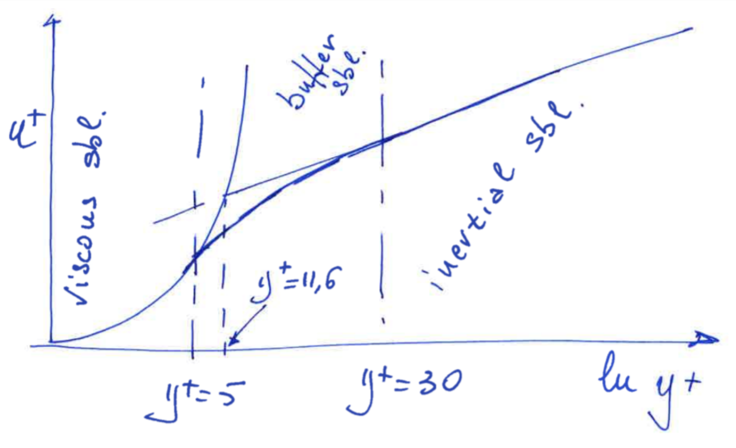
We have solved the equation in the following way:



3) Inertial sublayer: only the inertial term of the stress is important

$$u^+ = 2.5 \ln y^+ + 5.5$$

We can now try to plot the entire velocity profile:



Blasius profile

The law of the wall as derived in the logarithmic form is sometimes not used, preferring other empirical expressions, as the Blasius profile:

$$\boxed{\frac{\bar{v}_z(y)}{\bar{v}_{z,\max}}} = \left(\frac{y}{R}\right)^m$$

or

$$\boxed{\frac{\bar{v}_z(y)}{\langle \bar{v}_z \rangle} = \frac{(m+1)(m+2)}{2}\left(\frac{y}{R}\right)^m}$$

with $\frac{1}{10} < m < \frac{1}{6}$ (usually $m = \frac{1}{7}$).

With this equation, it is not possible to predict the shear stress at the wall: it would turn out to be infinite.
This equation is used with the famous Blasius relation to predict the friction factor in turbulent pipe flow:

$$\Delta P = 2\rho \langle v \rangle^2 f \frac{L}{D}$$

$$f = 0.079 Re^{-1/4}$$

or

$$\Delta P = \frac{1}{2}\rho \langle v \rangle^2 f \frac{L}{D}$$

$$f = 0.316 Re^{-1/4}$$