Modelling of Turbulent Flows

Lecture 4.1

April 26, 2020

Turbulence: Phenomenology & Modeling

Governing equations

We examined the different terms of the balance equations to identify one single scaling parameter (the Reynolds number, *Re*) and have the possibility to neglect some terms of the balance equations:

$$\textit{Continuity:} \quad \frac{\partial v_i}{\partial x_i} = 0$$

$$\textit{N-S:} \quad \rho \left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial {x_j}^2}$$

Simplifications

1. Creeping Flow:

   $$Re = \frac{\rho\, V\, D}{\mu} \to 0$$

   Inertia forces vanishing independent of the geometry.
2. Lubrication Approximation:

   Inertia forces negligible but with the help of a particular geometry.
3. Potential Flow:

   $$Re \to \infty$$

   Viscous forces *truly* negligible.
4. Boundary Layer Theory:

   All terms are to be considered: pressure may be estimated by the potential flow theory applied in the outer flow region.

Terms of N-S

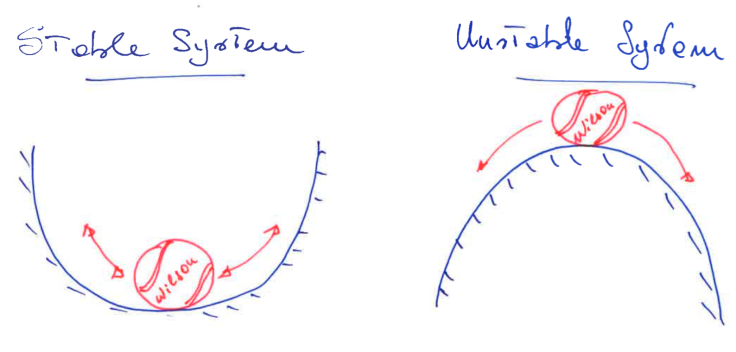
Look at the N-S equations from another viewpoint:

$$\underbrace{\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)}_{inertial\ \ terms} = \underbrace{-\frac{\partial P}{\partial x_i}}_{pressure\ \ term} + \underbrace{\mu\frac{\partial^2 v_i}{\partial {x_j}^2}}_{viscous\ \ term}$$

where the viscous term is also the term which dissipates.
The pressure term feeds energy into the flow system and inertial terms increase (so *there is acceleration*) but the viscous terms brake strongly with the viscous damping.

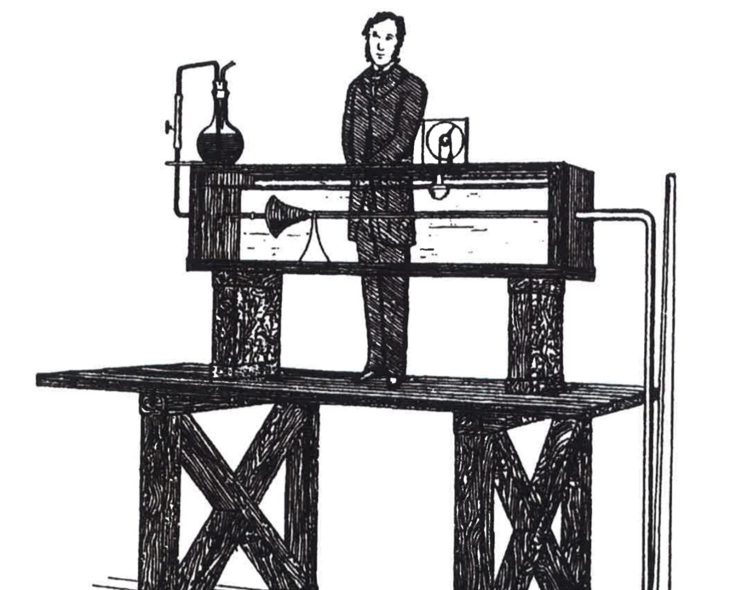
Stability

The Navier-Stokes equations describe a stable system if the inertial terms are not too big compared with the viscous terms. If they become too large, then the equations describe an unstable system. If the system is stable/unstable, then it is stable/unstable for arbitrarily big/small amplitudes of the perturbation.



Reynolds flow visualization

The experiment took place in Manchester, U.K., 1883.



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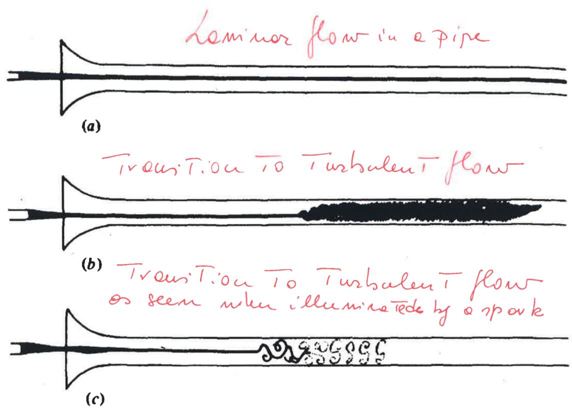
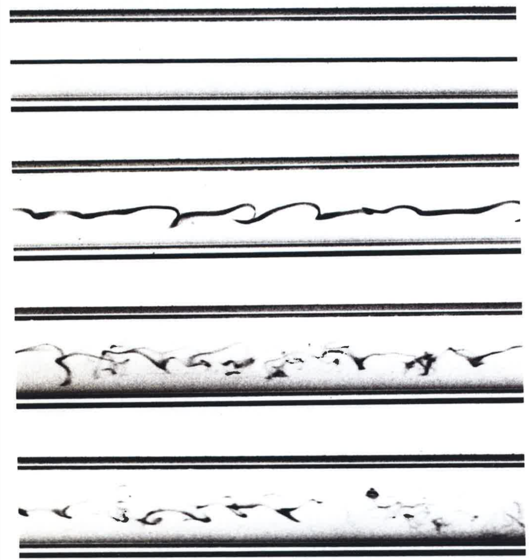


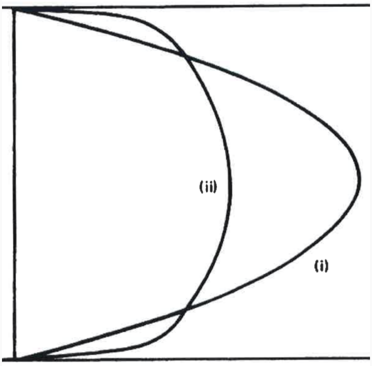
Fig. 9.2. Reynolds's drawings of the flow in his dye experiment.

Reynolds documented his experiment with sketches rather than photography. However, the original apparatus has survived and, a century later, N. H. Johannesen and C. Lowe have taken this sequence of photographs.



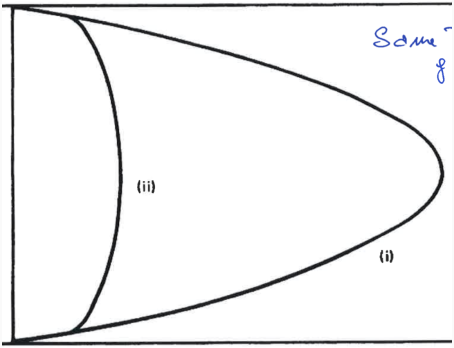
Macroscopic consequences of turbulence

The image shows a comparison of laminar (i) and turbulent (ii) velocity profiles in a pipe. The Reynolds number is about 4000.
Same velocity:



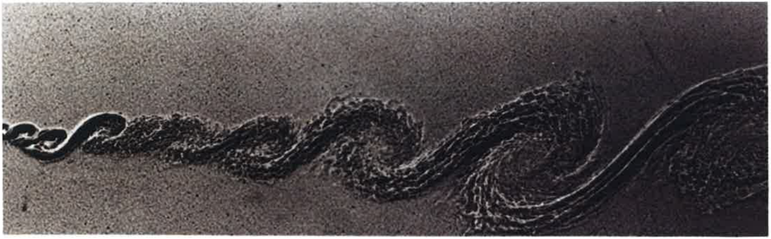
Macroscopic consequences of turbulence

The image shows a comparison of laminar (i) and turbulent (ii) velocity profiles in a pipe. The Reynolds number is about 4000.
Same pressure gradient:



Large-scale and small-scale structures in turbulence

Increasing the Reynolds number produces smaller structures:



176. Large-scale structure in a turbulent mixing layer. Nitrogen above flowing at 1000 cm/s mixes with a helium-argon mixture below at the same density flowing at 380 cm/s under a pressure of 4 atmospheres. Spark shadow photography shows simultaneous edge and plan views, demonstrating the spanwise organization of the large eddies. The streamwise streaks in the plan view (of which half the span is shown) correspond to a system of secondary vortex pairs oriented in the streamwise direction. Their spacing at the downstream side of the layer is larger than near the beginning. *Photograph by J. H. Konrad, Ph.D. thesis, Calif. Inst. of Tech., 1976.*

Large-scale and small-scale structures in turbulence

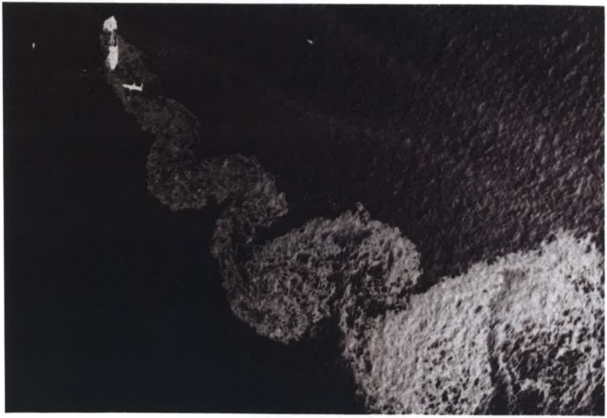
Increasing the Reynolds number produces smaller structures:

177. Coherent structure at higher Reynolds number. This flow is as above but at twice the pressure. Doubling the Reynolds number has produced more small-scale structure without significantly altering the large-scale structure. *M. R. Rebollo, Ph.D. thesis, Calif. Inst. of Tech., 1976; Brown & Roshko 1974*

Turbulent flow similarity

172. Wake of an inclined flat plate. The wake behind a plate at 45° angle of attack is turbulent at a Reynolds number of 4300. Aluminum flakes suspended in water show its characteristic sinuous form. *Cantwell 1981. Reproduced, with permission, from the* Annual Review of Fluid Mechanics, *Volume 13. © 1981 by Annual Reviews Inc.*

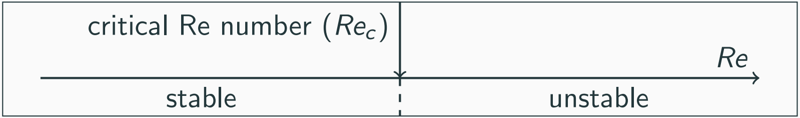
Turbulent flow similarity



173. Wake of a grounded tankship. The tanker *Argo Merchant* went aground on the Nantucket shoals in 1976. Leaking crude oil shows that she happened to be inclined at about 45° to the current. Although the Reynolds number is approximately 10^7, the wake pattern is remarkably similar to that in the photograph at the top of the page. *NASA photograph, courtesy of* O. M. Griffin, *Naval Research Laboratory.*

Observations

From the experiment of Reynolds we see that:

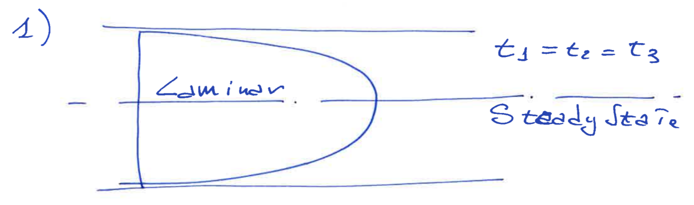


Below Re_c, perturbations are damped.

Above Re_c, perturbations are amplified.

Observations

Suppose we have a Poiseuille flow in a channel (flow driven by a pressure gradient).



Observations

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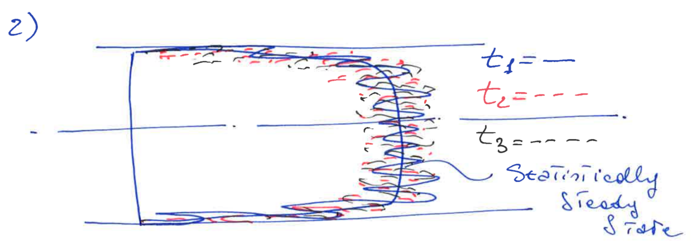
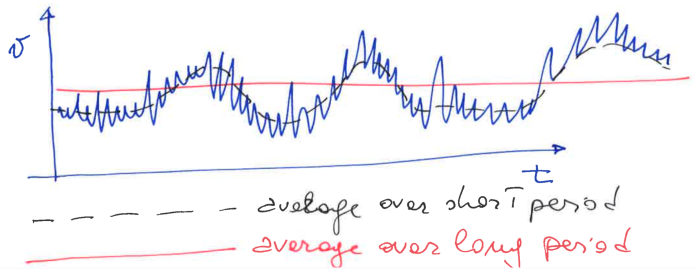


Observations

In fact, we are not really interested in the instantaneous behavior of the quantities (velocity, pressure, etc.), but rather in average quantities. A velocity probe in our channel will yield a signal like this:



Average of a variable

In the discrete space, the average of a generic variable ξ_i is:

$$\bar{\xi} = \frac{1}{N} \sum_{i=1}^{N} \xi_i$$

and in the continuous space

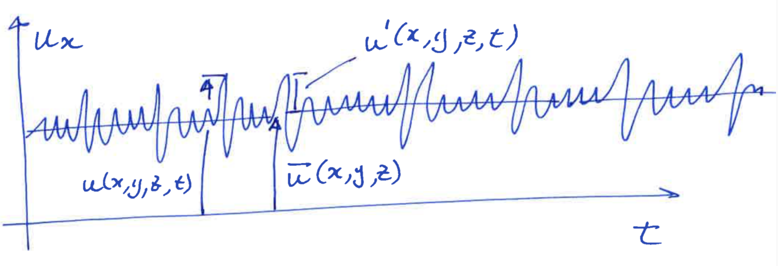
$$\bar{\xi}(t) = \frac{1}{2T} \int_{t-T}^{t+T} \xi(t) \mathrm{d}t$$

with $T \to \infty$.

Average of velocity

In a channel with constant pressure drop we have for the average velocity:

$$\bar{u}_i(x,y,z) = \lim_{t\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} u_i(x,y,z,t)\mathrm{d}t$$



Decomposition of velocity

We can decompose the velocity as:

$$u(x, y, z, t) = \bar{u}(x, y, z) + u'(x, y, z, t)$$

with:

- u: instantaneous velocity
- $\bar{u}$: average velocity **not** dependent on time
- u': fluctuating velocity

Decomposition of velocity

In general applications, we are not interested in the fluctuating part of the variables: we are rather interested in their average value. In the same way reasoned Reynolds, who proposed the following procedure:

1 We decompose the variables into average and fluctuating parts and we substitute them into the balance equations;

2 we average the equations over time so to obtain time independent equations in which most (hopefully all) fluctuating terms are eliminated;

3 we can solve these equations to obtain the average variables.

Application to balance equations

Balance equations:

$$\textit{Continuity:}\quad \frac{\partial v_i}{\partial x_i} = 0$$

$$\textit{N-S:}\quad \rho\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right] = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial {x_j}^2}$$

Variables:

$$v_i = \bar{v}_i + v_i' \;\;;\quad P = \bar{P} + P'$$

Mean Continuity equation

We start with the Continuity equation:

$$\frac{\partial}{\partial x_i}(\bar{v}_i + v_i') = \frac{\partial \bar{v}_i}{\partial x_i} + \frac{\partial v_i'}{\partial x_i} = 0$$

Applying the time average, we have (x and t are independent):

$$\overline{\frac{\partial \bar{v}_i}{\partial x_i}} = \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \frac{\partial \bar{v}_i}{\partial x_i} \mathrm{d}t = \frac{\partial \bar{v}_i}{\partial x_i}$$

$$\overline{\frac{\partial v_i'}{\partial x_i}} = \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \frac{\partial v_i'}{\partial x_i} \mathrm{d}t =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \frac{\partial}{\partial x_i} \underbrace{\int\limits_{-T}^{T} v_i' \mathrm{d}t}_{=0 \text{ by definition}} = 0$$

Mean Continuity equation

The Continuity equation becomes

$$\boxed{\frac{\partial \bar{v}_i}{\partial x_i} = 0}$$

which, substituted in the non-averaged Continuity equation, gives

$$\frac{\partial v_i'}{\partial x_i} = 0$$

So we have that **both** the divergence of the average velocity and the divergence of the fluctuating velocity are zero (representing average Continuity and fluctuating Continuity, respectively).

Mean N-S equations

Using the same procedure with the N-S equations, we can write:

$$\rho\left[\frac{\partial(\bar{v}_i+v_i')}{\partial t}+(\bar{v}_j+v_j')\frac{\partial(\bar{v}_i+v_i')}{\partial x_j}\right]=-\frac{\partial(\bar{P}+P')}{\partial x_i}+\mu\frac{\partial^2(\bar{v}_i+v_i')}{\partial {x_j}^2}$$

Then we take the average

$$\rho\left[\overline{\frac{\partial(\bar{v}_i+v_i')}{\partial t}}+\overline{(\bar{v}_j+v_j')\frac{\partial(\bar{v}_i+v_i')}{\partial x_j}}\right]=-\overline{\frac{\partial(\bar{P}+P')}{\partial x_i}}+\mu\overline{\frac{\partial^2(\bar{v}_i+v_i')}{\partial {x_j}^2}}$$

$$\rho\left[\overline{\frac{\partial\bar{v}_i}{\partial t}}+\overline{\frac{\partial v_i'}{\partial t}}+\overline{\bar{v}_j\frac{\partial\bar{v}_i}{\partial x_j}}+\overline{\bar{v}_j\frac{\partial v_i'}{\partial x_j}}+\overline{v_j'\frac{\partial\bar{v}_i}{\partial x_j}}+\overline{v_j'\frac{\partial v_i'}{\partial x_j}}\right]=$$

$$=-\overline{\frac{\partial\bar{P}}{\partial x_i}}-\overline{\frac{\partial P'}{\partial x_i}}+\mu\overline{\frac{\partial^2\bar{v}_i}{\partial {x_j}^2}}+\mu\overline{\frac{\partial^2 v_i'}{\partial {x_j}^2}}$$

Mean N-S equations

Knowing that:

- time and space are independent
- the average of an average is the average itself
- the following identities hold

$$\overline{\frac{\partial \bar{v}_i}{\partial t}} = 0 \ \ ; \quad \overline{\frac{\partial v_i'}{\partial t}} = 0 \ \ ; \quad \overline{\frac{\partial p'}{\partial x_i}} = 0$$

$$\overline{\bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j}} = \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j}$$

$$\overline{\bar{v}_j \frac{\partial v_i'}{\partial x_j}} = 0 \ \ ; \quad \overline{v_j' \frac{\partial \bar{v}_i}{\partial x_j}} = 0$$

$$\overline{v_j' \frac{\partial v_i'}{\partial x_j}} = \overline{\frac{\partial v_j' v_i'}{\partial x_j}} - \overline{v_i' \cancel{\frac{\partial v_j'}{\partial x_j}}} = \overline{\frac{\partial v_j' v_i'}{\partial x_j}} \ \text{ since } \ \frac{\partial v_j'}{\partial x_j} = 0$$

$$\overline{\frac{\partial^2 v_i'}{\partial {x_j}^2}} = 0$$

Mean N-S equations

Substituting into the N-S equations we have:

$$\boxed{\rho\left[\bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j}\right] = -\frac{\partial \bar{P}}{\partial x_i} + \mu\frac{\partial^2 \bar{v}_i}{\partial {x_j}^2} - \rho\overline{\frac{\partial v'_j v'_i}{\partial x_j}}}$$

So we have an equation with averaged variables but with an **extra term** too:

$$\rho\left[\bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j}\right] = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\frac{\partial \bar{v}_i}{\partial x_j}\right] - \frac{\partial}{\partial x_j}\left[\rho\overline{v'_j v'_i}\right]$$

$$\rho\left[\bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j}\right] = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\underbrace{\mu\frac{\partial \bar{v}_i}{\partial x_j}}_{\tau_{ij}} - \underbrace{\rho\overline{v'_j v'_i}}_{\bar{\tau}^R_{ij}}\right]$$

$\bar{\tau}^R_{ij}$ is the Reynolds stress tensor. The elements of this tensor are the so-called Reynolds stresses, $\rho\overline{v'_j v'_i}$.

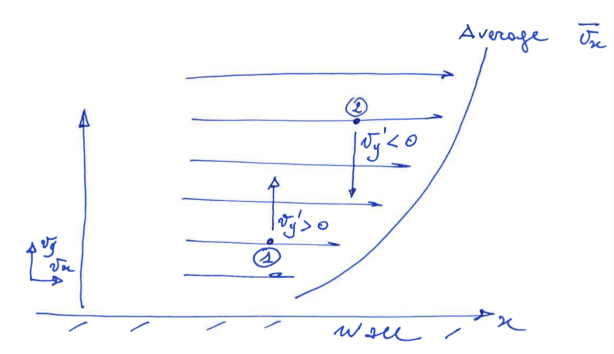
Recap

We have been able to derive an average equation in which only average variables appear. However, we have now the problem of getting acquainted with the new character: $\overline{\tau}_{ij}^{R}$.
As the equation is now cast, it seems that we have more damping for the same pressure gradient. And if it were true, that would explain the velocity profiles we have seen before: in turbulent flows we are able to transfer less flowrate for the same pressure gradient.

Reynolds stresses

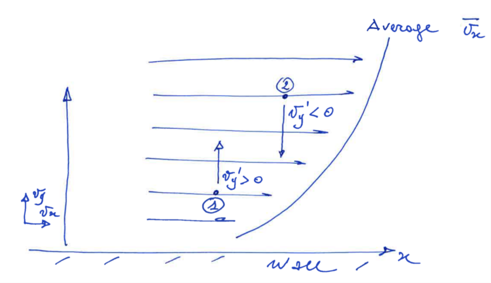
$\bar{\tau}_{ij}^R = -\rho\overline{v_j' v_i'} \neq 0$ *only if* the two variables, v_i' and v_j', are not independent of each other.

We observe the situation of the turbulent channel flow to examine the behavior of the Reynolds' stresses.



Note: $i \equiv x$, $j \equiv y$.

Reynolds stresses

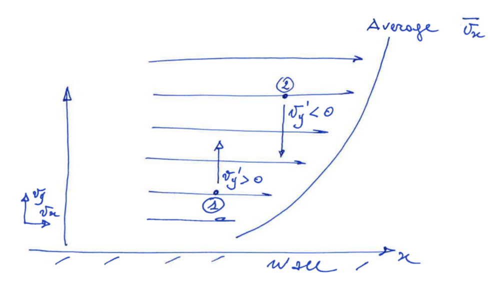


The fluid parcel ① has a positive velocity fluctuation in the y direction (average $\bar{v}_y = 0$) and moves in a region where all parcels have a larger velocity in the x direction.

It is likely (we speak of statistical fluctuations) that the instantaneous velocity in x will be lower than the mean: $v_x' < 0$

$$\Rightarrow v_y' > 0 \text{ and } v_x' < 0 \Rightarrow v_x' v_y' < 0$$

Reynolds stresses



The fluid parcel ② has a negative velocity fluctuation in the y direction.

It is likely that the instantaneous velocity in x will be higher than the mean: $v_x' > 0$

$$\Rightarrow v_y' < 0 \text{ and } v_x' > 0 \Rightarrow v_x' v_y' < 0$$

Then, for these two parcels:

$$\bar{\tau}_{yx} = \mu \frac{\partial \bar{v}_x}{\partial y} - \rho \overline{v_x' v_y'} > \mu \frac{\partial \bar{v}_x}{\partial y}$$