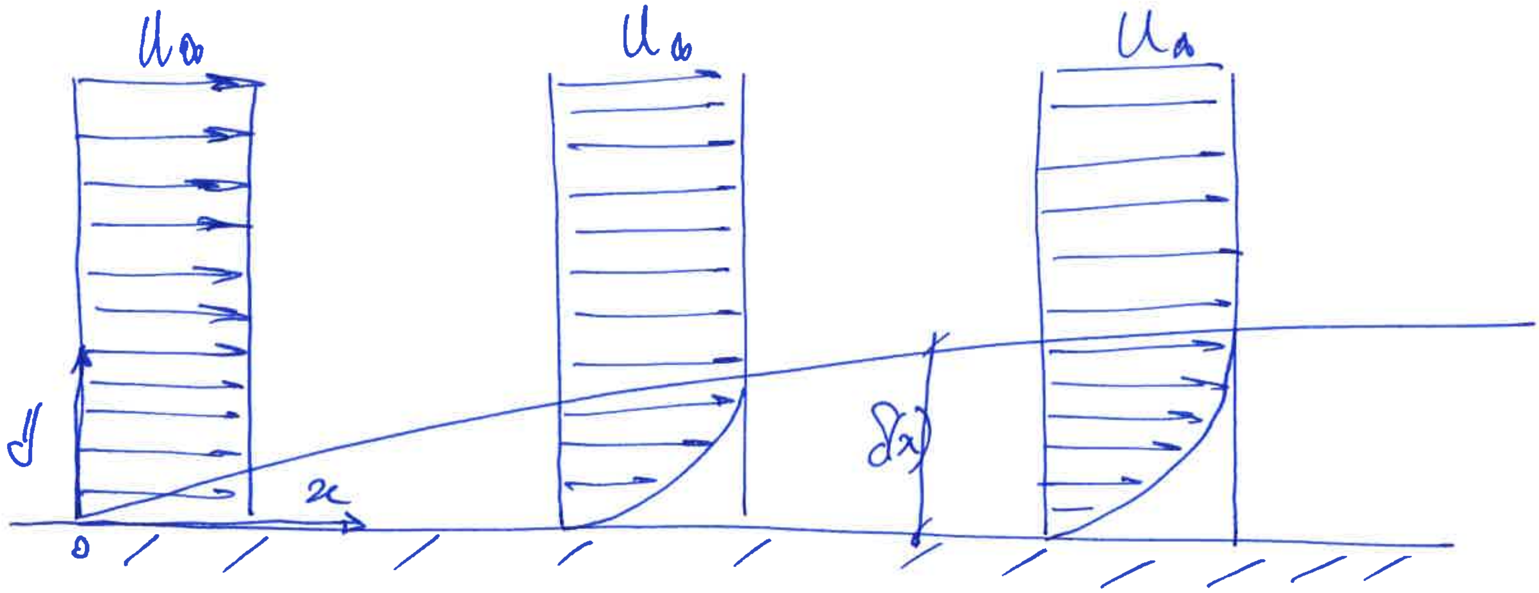
Vorlesung 9

BOUNDARY LAYER ON A FLAT PLATE — BLASIUS B. L.



The available equations are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

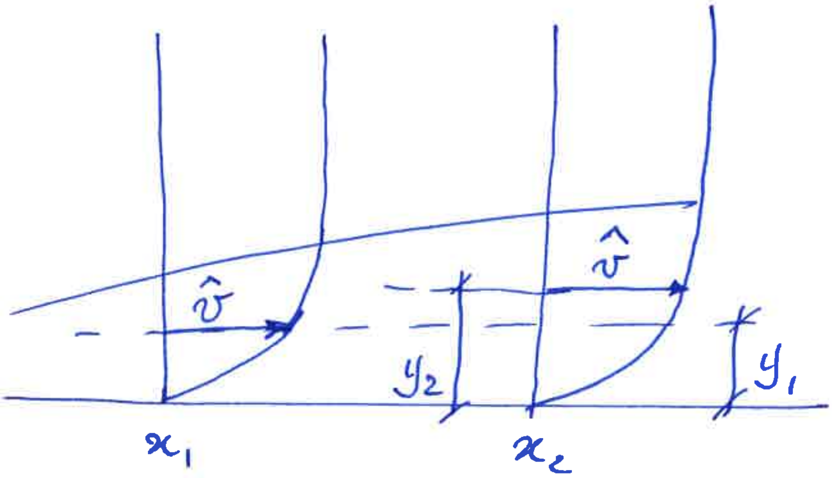
$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = \rho U_\infty \frac{dU_\infty}{dx} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

Since U_∞ is constant, $\dfrac{dU_\infty}{dx} = 0$.

The problem in this case is rather complicate: we have two unknown dependent variables, v_x and v_y, which both depend on the two independent variables, x and y.

$$v_x(x,y) \neq 0 \quad \text{and} \quad v_y(x,y) \neq 0$$

We can apply the similarity theory to reduce the dependence on two variables to the dependence on one function of two variables.



We find the same value of the velocity $\hat{v}$ at two different stations x_1 and x_2 for two different values of y. So we can identify a <u>similarity</u> function, $\eta(x,y)$ so that

$$\hat{v}_x(\hat{\eta}) = \hat{v}[\hat{\eta}(x_1,y_1)] = \hat{v}[\hat{\eta}(x_2,y_2)]$$

(*)

NOTE →

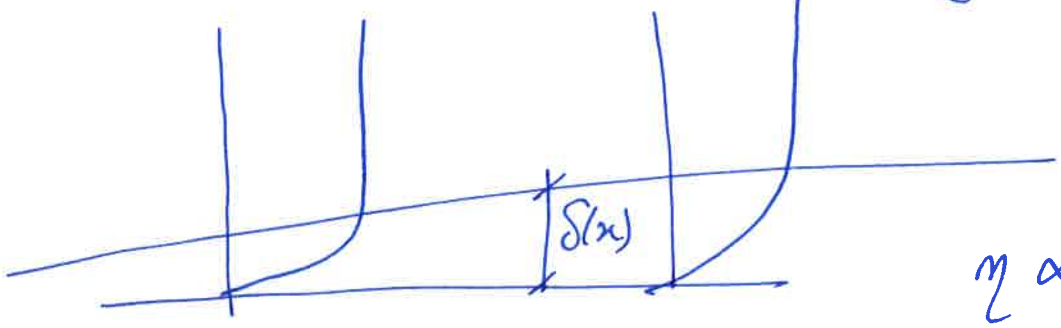
However, in this case this is not enough since we still have v_x and v_y.
If we ~~xxxxx~~ use the streamfunction we can however reduce the unknown variables to one. This at the price of increasing the order of the differential equation.

Eqs.

$$\begin{cases} \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0 \\ v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y} = \nu \dfrac{\partial^2 v_x}{\partial y^2} \end{cases}$$

BCs $v_x = v_y = 0$ @ $y = 0$; $v_x = U_\infty$ as $y \to \infty$

⊛ Note: The similarity function is actually a similarity variable obtained just by rescaling the y coordinate with the thickness of the Boundary Layer.



$\eta \propto y/\delta(x)$

The concept is the STRETCHING of the coordinate

Streamfunction

$$v_x = -\frac{\partial \psi}{\partial y} \quad ; \quad v_y = \frac{\partial \psi}{\partial x}$$

Continuity

$$-\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

Automatically satisfied

$NS)_x$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\nu \frac{\partial^3 \psi}{\partial y^3}$$

Defining the similarity variable η as

$$\eta = y/\delta(x) = y\Big/\sqrt{\frac{\nu x}{U_\infty}} = y\sqrt{\frac{U_\infty}{\nu x}}$$

Computing v_x from the definition of ψ:

$$\psi = f(\eta) \Longrightarrow v_x = -\frac{\partial \psi}{\partial y} = -\frac{\partial f(\eta)}{\partial y} =$$

$$= -\frac{df}{d\eta}\frac{\partial \eta}{\partial y} = -f'\sqrt{\frac{U_\infty}{\nu x}}$$

We have to avoid an explicit dependence of v_x on x. In addition we would like

To "fix" the dimensions of the streamfunction and (5)
so we would rather have a dimensionless $f(\eta)$.

So we define: $$\boxed{\psi(\eta) = -\sqrt{\nu u_\infty x}\, f(\eta)}$$
where f is
dimensionless and $[\psi] = m^2/s$

We thus find that $$\boxed{v_x = -\frac{\partial \psi}{\partial y} = u_\infty f'}$$

We need now to substitute ψ into the Navier-Stokes equation expressed in terms of ψ.

- $\cancel{\frac{\partial \psi}{\partial y} = + \sqrt{\nu u_\infty x}\, \frac{df}{d\eta}}$

- $$\boxed{\frac{\partial \psi}{\partial y} = - v_x = - u_\infty f'}$$ (1)

- $$\boxed{\frac{\partial \psi}{\partial x} = v_y} = -\frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\, f - \sqrt{\nu u_\infty x}\, \frac{df}{d\eta}\frac{\partial \eta}{\partial x} =$$

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$$= -\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\, f - \sqrt{\nu U_\infty x}\, f' \cdot \left(-\frac{1}{2}\right)\underbrace{\frac{y}{x}\sqrt{\frac{U_\infty}{\nu x}}}_{=\,\eta} =$$

$$\boxed{= -\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\, f + \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\, \eta f'} \qquad (2)$$

$$\bullet\ \boxed{\frac{\partial^2 \psi}{\partial x \partial y}} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial x}\left[-U_\infty f'\right] =$$

$$= -U_\infty \frac{df'}{d\eta}\frac{\partial \eta}{\partial x} = -U_\infty f''\left(-\frac{1}{2}\right)\frac{y}{x}\sqrt{\frac{U_\infty}{\nu x}} =$$

$$\boxed{= \frac{1}{2}\frac{U_\infty}{x}\eta f''} \qquad (3)$$

$$\bullet\ \boxed{\frac{\partial^2 \psi}{\partial y^2}} = -U_\infty f''\frac{\partial \eta}{\partial y} = \boxed{-U_\infty\sqrt{\frac{U_\infty}{\nu x}}\, f''} \qquad (4)$$

$$\bullet\ \boxed{\frac{\partial^3 \psi}{\partial y^3}} = -U_\infty\sqrt{\frac{U_\infty}{\nu x}}\, f'''\frac{\partial \eta}{\partial y} = \boxed{-\frac{U_\infty^2}{\nu x} f'''} \qquad (5)$$

Substituting into the N/S$)_x$

$$-U_\infty f' \cdot \frac{1}{2}\frac{U_\infty}{x}\eta f'' + \left[\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\, f - \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\,\eta f'\right] \cdot \left(-U_\infty\sqrt{\frac{U_\infty}{\nu x}}\, f''\right) = +\frac{\nu U_\infty^2}{\nu x} f'''$$

$$\Rightarrow -\frac{U_\infty^2}{2x}\cancel{\eta f' f''} + \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[f - \cancel{\eta f'}\right]\cdot\left(-U_\infty\sqrt{\frac{U_\infty}{\nu x}}\, f''\right) = +\frac{\nu U_\infty^2}{\nu x} f'''$$

$$\Rightarrow -\frac{1}{2}\frac{U_\infty^2}{x} f f'' = \frac{U_\infty^2}{x} f'''$$

$$\Rightarrow \boxed{f''' + \frac{1}{2} f f''}$$

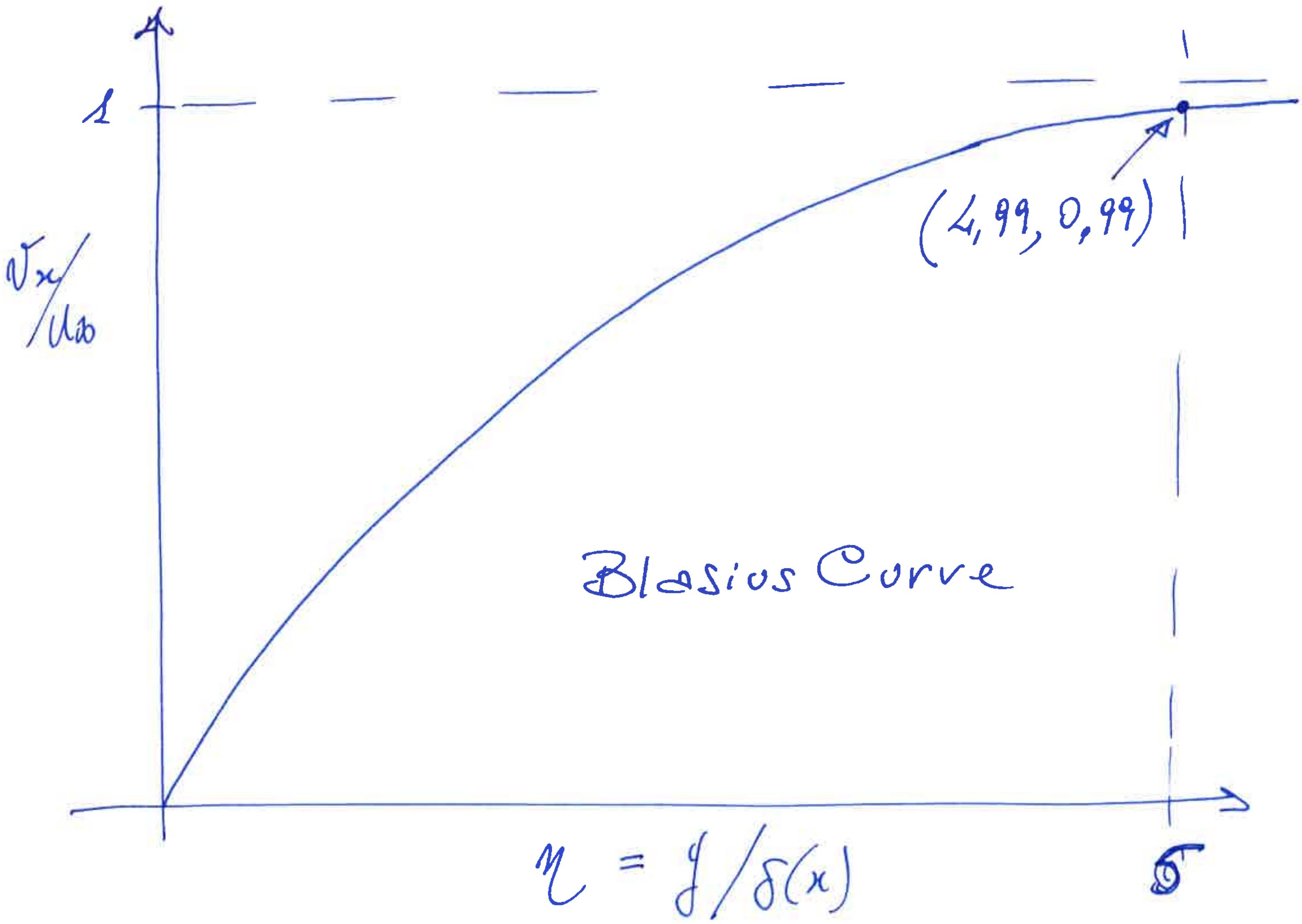
With Boundary conditions

$$\begin{cases} f' = 0 & @\ \eta = 0 \\ f' = 1 & @\ \eta \to \infty \\ f = 0 & @\ \eta = 0 \end{cases}$$

The equation to be solved is:

$$\begin{cases} \eta = y\sqrt{\dfrac{U_\infty}{\nu x}} = y/\delta(x) \\ f''' + \frac{1}{2} f f'' = 0 \end{cases}$$

Unfortunately, this equation, although coming from an elegant derivation, must be found by numerical integration.



Wall shear stress

We started all this to compute the forces acting at the wall.

Wall shear stress is:

$$\tau_w(x) = \mu \frac{\partial v_x}{\partial y}\Bigg|_{y=0} = -\mu \frac{\partial^2 \psi}{\partial y^2}\Bigg|_{y=0} =$$

$$= \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}}\, f''\Big|_{y=0} =$$

$$= \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}}\, f''\Big|_{y=0}$$

Numerically $f''\big|_{y=0} = f''(0) = 0,332$

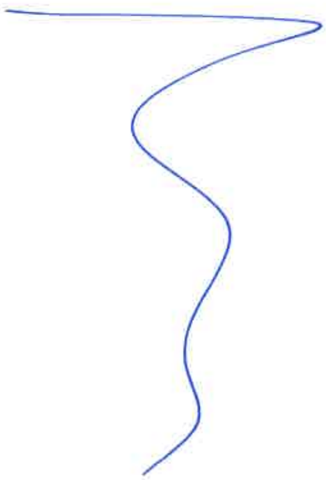
$$\boxed{\tau_w(x) = 0,332\, U_\infty \mu \sqrt{\frac{U_\infty}{\nu x}}}$$

And the thickness of the B.L.

$$\boxed{\delta^* = \int_0^\infty \left[1 - \frac{v_x(y)}{U_\infty}\right] dy = 1,72 \sqrt{\frac{\nu x}{U_\infty}}}$$

The Boundary Layer Thickness increases with $\sqrt{x}$.

The wall shear stress decreases with $\frac{1}{\sqrt{x}}$.
This is expected given the decreasing slope of the velocity profile with "x".



If we consider The cylinder example :

- In (A) $\quad \frac{\partial P}{\partial x} < 0 \qquad \frac{\partial u_x}{\partial y} > 0$

$$\tau_w = \mu \frac{\partial v_x}{\partial y} > 0 \quad ; \quad \omega_z = -\frac{\partial u_x}{\partial y} < 0$$

[Clockwise rotation]

- In (S) $\quad \frac{\partial P}{\partial x} = 0 \qquad \frac{\partial u_x}{\partial y} = 0$

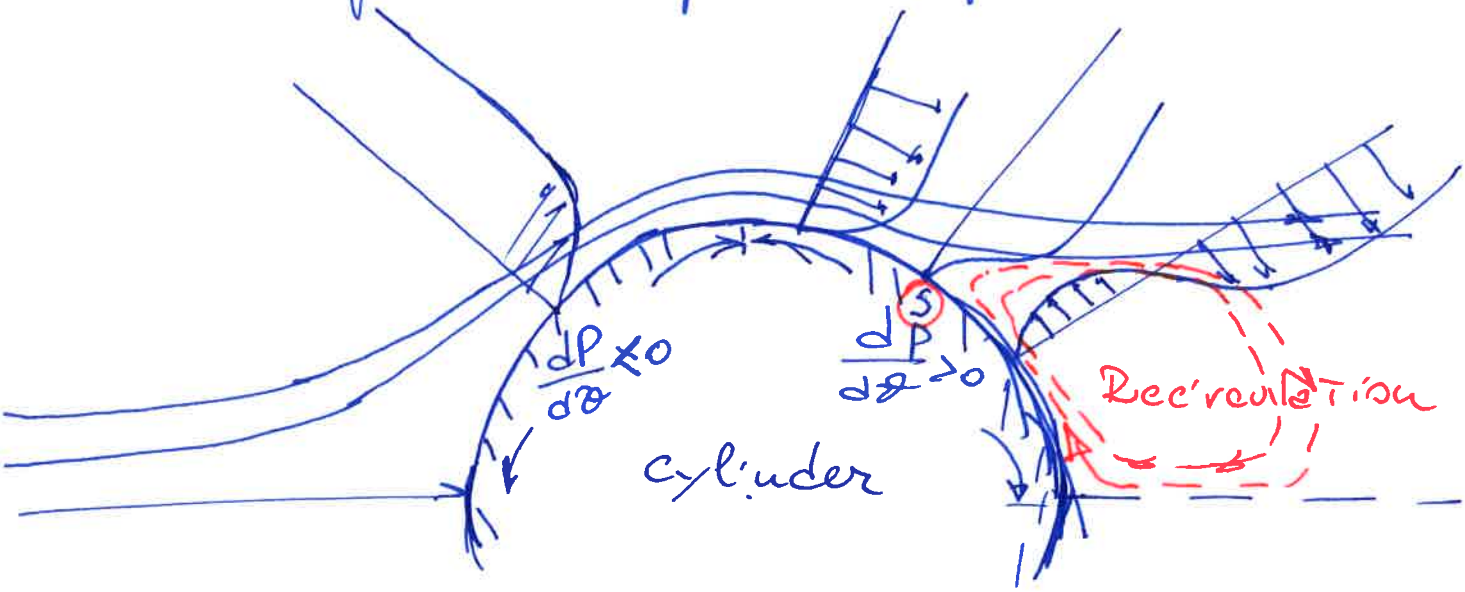
$$\tau_w = \mu \frac{\partial u_x}{\partial y} = 0 \quad ; \quad \omega_z = 0$$

- In (B) $\quad \frac{\partial P}{\partial x} > 0 \qquad \frac{\partial u_x}{\partial y} < 0$

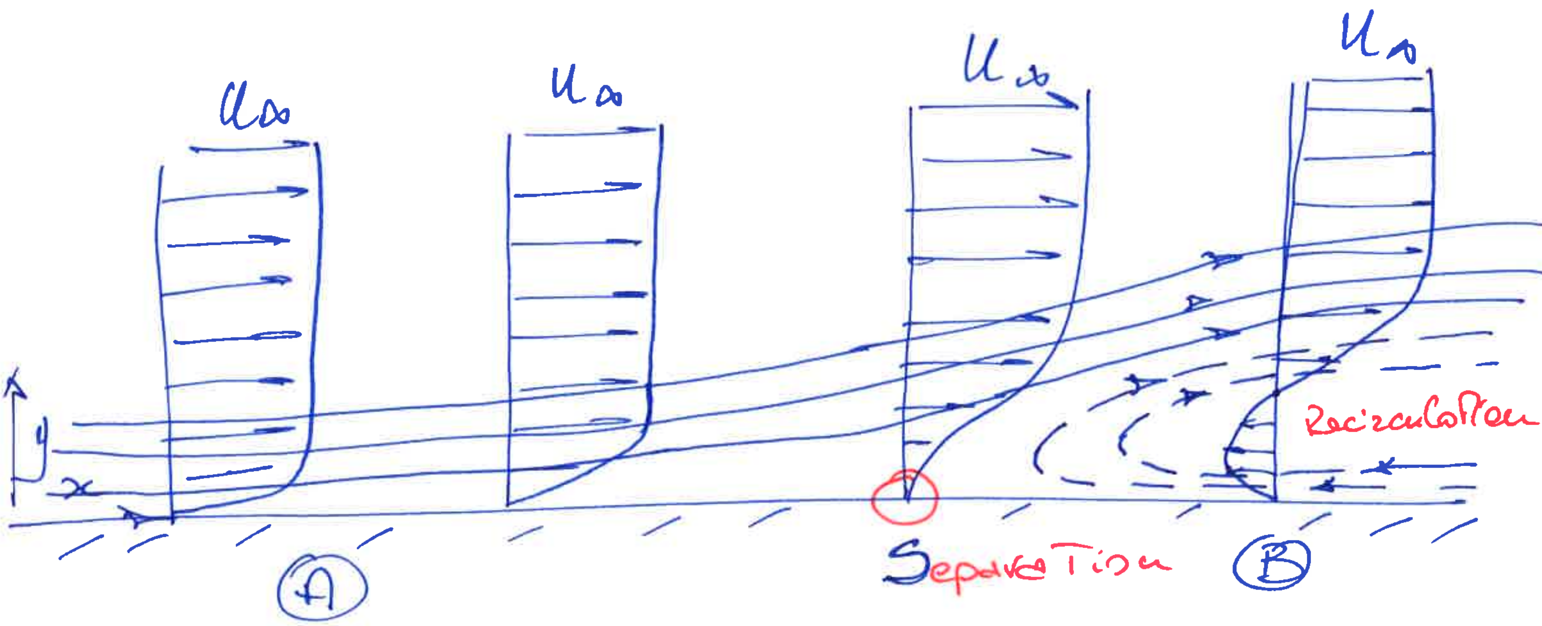
$$\tau_w < 0 \quad ; \quad \omega_z > 0$$

Detachment of Boundary Layers Separation

It was not possible to predict the Drag on a Cylinder with potential flow theory because the Boundary layer detaches from the sphere surface.



We can "straighten" the sphere/cylinder surface on a plane.



Why the B.C. separates?

In case we have a flat plate the boundary layer may separate? May be.

The boundary layer thickness increases with "x" and the velocity gradient decreases.

If the velocity gradient reaches zero the velocity profile has a vertical derivative at the wall.

However, to change direction of the velocity we need that the second derivative of the velocity profile changes sign $\boxed{\left.\dfrac{\partial^2 v_x}{\partial y}\right|_{y=0}}$

Now, at the wall inertia is null and the N/S$)_x$ becomes.

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

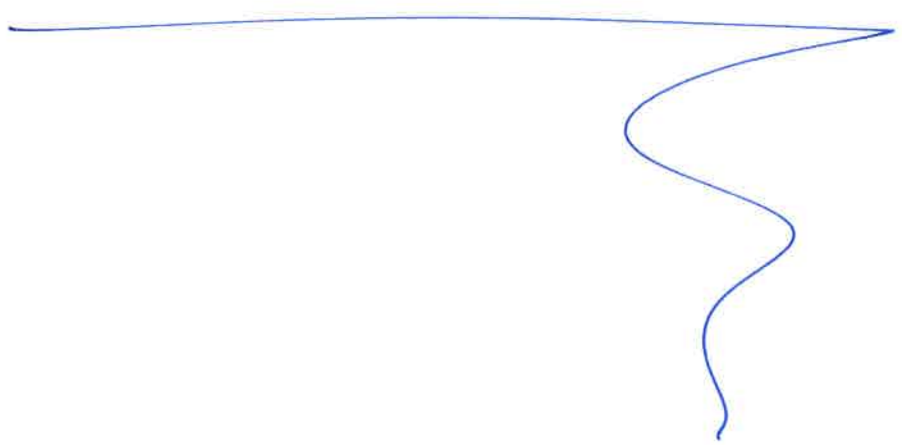
$$\left.\frac{\partial v_x}{\partial x}\right|_{y=0} = 0$$

Then:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}\Big|_{y=0}$$

and we remember that $\frac{\partial p}{\partial y} = 0$, so the pressure behavior at the wall depends on what happens in the external flow.

So IF the pressure gradient in the external flow does not change sign, the concavity of the velocity profile may not happen and the boundary layer does not separate.



Classification of Boundary Layers

So, what happens in the B.L. depends on the what happens in the outer flow, and in particular on the behavior of the equation:

$$\boxed{\frac{dp}{dx} = -\rho U_\infty \frac{dU_\infty}{dx}}$$

Accelerating Boundary Layer

$$\frac{dU_\infty}{dx} > 0 \qquad \frac{dp}{dx} < 0$$

The pressure decreases with "x" and we can describe it as B.L. in a favourable pressure gradient.
In this case the b.l. is thin and the advection dominates the viscous diffusion, which is the cause of expansion of the b.l. - This B.L. has similar dynamics to the zero pressure gradient b.l. which we examined in great detail

Decelerating Boundary Layer

$$\frac{du_a}{dx} < 0 \qquad \frac{dp}{dx} > 0$$

The pressure increases with x, and we speak of an Adverse pressure gradient. In this case, the advection cannot work much and the thickness of the B.C. increases. This boundary layer is prone to separation.