Modelling of Turbulent Flows

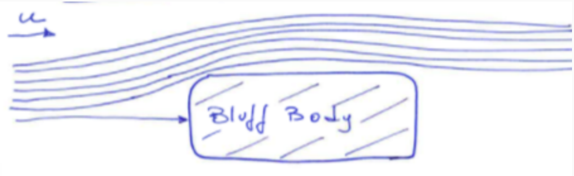
Lecture 3.1

April 17, 2020

Boundary Layers

Outline

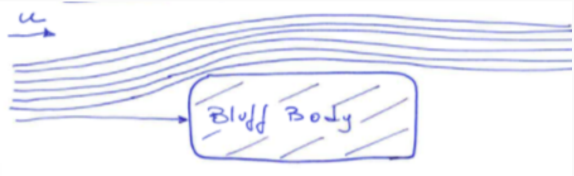
Boundary Layers



u is the free stream velocity.

When a fluid in uniform flow at a velocity u meets with a bluff bodythe speed at which disturbances (i.e. modifications to the flow) are transported downstream is u. However, the velocity at which disturbances are transported away from the wall (i.e. the velocity with which the information at the wall is transported to the outer flow) is proportional to the viscous diffusion velocity.

Boundary Layers

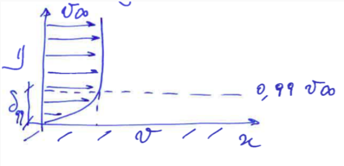


u is the free stream velocity.

The perception is thus that disturbances are generated at the wall but are also rapidly transported downstream, leaving the **far field** essentially undisturbed.

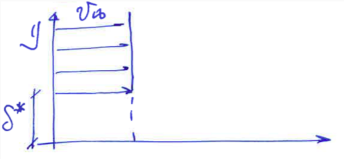
This is the concept of the **boundary layer**, which allows to confine the interactions of the body with fluid to a very thin region around the body.

Thickness of the B.L.



δ_{99} is the distance from the wall at which the velocity reaches 99% of the free stream velocity.

Displacement thickness



$\delta^\star$ is the thickness of a layer with 0 (zero) velocity which produces the same deficit of mass transported downstream.

$$v_\infty \delta^\star = \int_0^\infty v_\infty \mathrm{d}y - \int_0^\infty v(y)\mathrm{d}y = \int_0^\infty [v_\infty - v(y)]\mathrm{d}y \Rightarrow \delta^\star = \int_0^\infty \left[1 - \frac{v(y)}{v_\infty}\right] \mathrm{d}y$$

Momentum thickness

$\hat{\delta}$ is the thickness of a layer with zero velocity which produces the same deficit of momentum transport downstream.

$$\rho\,\hat{\delta}\,v_{\infty}^{\star 2} = \rho \int_0^{\infty} v(y)[v_{\infty} - v(y)]\mathrm{d}y$$

$$\Rightarrow \hat{\delta} = \int_0^{\infty} \frac{v(y)}{v_{\infty}} \left[1 - \frac{v(y)}{v_{\infty}}\right] \mathrm{d}y$$

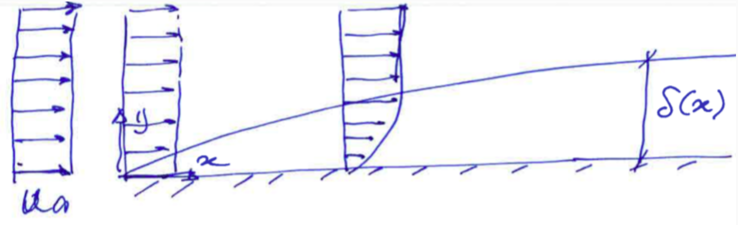
It holds:

$$\boxed{\delta_{99} > \delta^{\star} > \hat{\delta}}$$

Boundary Layer Equations

We want to derive the equations which are necessary to solve the velocity profile inside the boundary layer which forms when a free stream moving at velocity u_∞ meets a flat stationary plate.

We hypothesize steady state flow and 2D geometry as sketched:



Continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

N-S$_x$:

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

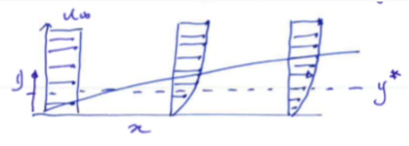
N-S$_y$:

$$\rho \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right)$$

We wish to make the equations dimensionless so to appreciate the order of magnitude of each term. In this case we have no characteristic length; we choose $\delta(x)$ for the y direction and L (the length of the plate) for the x direction. u_∞ and V will be the characteristic velocities. δ and V are unknown.

Continuity

$$\frac{u_\infty}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{V}{\delta}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$$



If we move along the line at $y^\star$, we observe that v_x changes and therefore it is $v_x(x, y)$.

Then both terms in the continuity equation must be of some order of magnitude and

$$\frac{u_\infty}{L} \cdot \frac{\delta}{V} = \mathcal{O}(1) \Rightarrow V \simeq \frac{\delta}{L} u_\infty$$

with $\frac{\delta}{L} \ll 1$ and consequently $\frac{v}{u_\infty} \ll 1$.

Navier-Stokes (x)

$$\rho\frac{\delta^2 u_\infty}{L\,\mu}\left[\tilde{v}_x\frac{\partial\tilde{v}_x}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\tilde{v}_x}{\partial\tilde{y}}\right]=-\frac{\Pi\,\delta^2}{\mu\,u_\infty\,L}\cdot\frac{\partial\tilde{P}}{\partial\tilde{x}}+\left[\left(\frac{\delta}{L}\right)^2\frac{\partial^2\tilde{v}_x}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{v}_x}{\partial\tilde{y}^2}\right]$$

with Π the scaling pressure.

We learned that by definition, in the Boundary Layer, inertial and viscous terms must have the same order of magnitude and then

$$\rho\frac{u_\infty\delta^2}{L\,\mu}=\mathcal{O}(1)\Rightarrow\delta\simeq\sqrt{\frac{\mu\,L}{\rho\,u_\infty}}$$

Navier-Stokes (x)

We can define a Reynolds number for the B.L. as

$$\boxed{Re_L := \frac{\rho\, L\, u_\infty}{\mu}}$$

and then

$$\boxed{\delta = L \cdot Re_L^{-1/2}}$$

and we find that pressure scales with inertial forces:

$$\Pi = \mu \frac{u_\infty\, L}{\delta^2} = \rho\, u_\infty^2$$

The diffusion term $\left(\frac{\delta}{L}\right)^2 \cdot \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}$ is negligible and we have

$$\text{N-S}_x\text{: } \rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial P}{\partial x} + \frac{\partial^2 v_x}{\partial y^2}$$

Navier-Stokes (y)

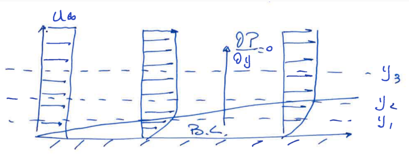
$$\left(\frac{\delta}{L}\right)^2 \left[\tilde{v}_x \frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_y}{\partial \tilde{y}}\right] = -\frac{\partial \tilde{P}}{\partial \tilde{y}} + \left(\frac{\delta}{L}\right)^2 \left[\left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}\right]$$

All terms except for the pressure gradient are negligible, thus:

$$\text{N-S}_y\text{: } \frac{\partial \tilde{P}}{\partial \tilde{y}} \simeq 0$$

which tells us that $\boxed{P = P(x)}$.

Pressure



Because the pressure does not depend on y, but only on x, we can determine the behavior of the pressure along y_3 with the Potential Flow equation for pressure (the Bernoulli equation). Assuming no gravity effect,

$$p - \frac{1}{2}\rho\, u_\infty^2 = \text{const}$$

$$\frac{\partial P}{\partial x} = \frac{\mathrm{d}P}{\mathrm{d}x} = -\rho\, u_\infty \frac{\mathrm{d}u_\infty}{\mathrm{d}x}$$

Boundary Layer equations

The equations for the BL are:

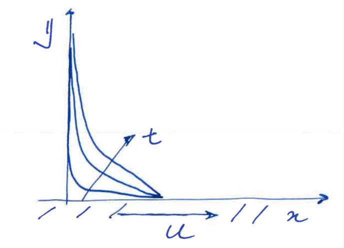
$$\text{continuity: } \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\text{N-S: } \rho \left[\underbrace{\frac{\partial v_x}{\partial t}}_{*} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right] = \rho\, u_\infty \frac{\mathrm{d} u_\infty}{\mathrm{d} x} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

($*$: this term only when we want to study non-stationary B.L.)

Stokes' Boundary Layer

(We solve this problem first, to become familiar with the similarity procedure) We consider a flat plate, infinitely long, which is suddenly set in motion.



The plate is aligned with the x axis and the half infinite domain above is fluid which is still. At $t = 0$ the plate is set in motion in the x direction at velocity u.

The half domain is characterized by a velocity directed only along x:

$$v_x \neq 0 \text{ and } v_y = 0$$

Of course, in this case v_x is not function of x, but only of y and t.

Continuity:

$$\frac{\partial v_x}{\partial x} + \cancel{\frac{\partial v_y}{\partial y}} = 0$$

N-S:

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

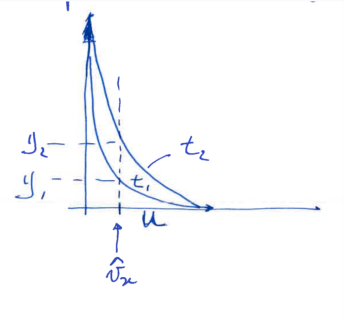
Boundary conditions:

$$v_x(0, t) = \begin{cases} 0 & t \leq 0 \\ u & t > 0 \end{cases} \quad ; \quad v_x(y \to \infty, t) = 0$$

In this problem there is no advection of momentum, which is transferred along y just by the diffusion term. It is analogous to heat transfer from a wall or mass transfer, for instance CO_2 absorption at a film surface.

The solution is based on the **similarity theory**.

The differential equation has one dependent variable which is a function of two independent variables (y and t).



We see that $\hat{v}_x(y_1, t_1) = \hat{v}_x(y_2, t_2)$, so we look for a function η of y and t which has the following form:

$$\eta(y, t) = \alpha \frac{y}{t^n}$$

with α and n to be determined.

For the similarity theory, it is:

$$\frac{v_x(y, t)}{u} = f(\eta)$$

On an intuitive basis this is justified with the following:
If u is doubled, then we expect v_x to double, so v_x is proportional to u and its slope is given by $f(\eta)$.
Our problem is now:

$$\begin{cases} v_x(y,t) = u \cdot f(\eta) & \text{Ⓐ} \\ \underbrace{\frac{\partial v_x}{\partial t}}_{I} = \nu \underbrace{\frac{\partial^2 v_x}{\partial y^2}}_{II} & \text{Ⓑ} \end{cases}$$

Substituting Ⓐ in Ⓑ, the term I becomes

$$\frac{\partial v_x}{\partial t} = u \frac{\mathrm{d}f}{\mathrm{d}\eta} \frac{\partial \eta}{\partial t} = f' \alpha (-n) \frac{y}{t^{n+1}} u = -\frac{n}{t} \eta f' u$$

and the term II is

$$\frac{\partial v_x}{\partial y} = u f' \frac{\partial \eta}{\partial y} = u f' \frac{\alpha}{t^n}$$

$$\frac{\partial^2 v_x}{\partial y^2} = u \frac{\alpha^2}{t^{2n}} f''$$

Substituting,

$$-\frac{u}{t}\eta f'\cancel{u} = \cancel{u}\frac{\alpha^2}{t^{2n}}f''\nu$$

We do not want an explicit dependence on t and therefore we choose $n = 1/2$:

$$\boxed{\nu\alpha^2 f'' = -\frac{\eta}{2}f'}$$

in which $\eta = \frac{\alpha y}{\sqrt{t}}$.
Since η must be dimensionless, we choose

$$\alpha = \frac{1}{2\sqrt{\nu}} \Rightarrow \eta = \frac{1}{2}\frac{y}{\sqrt{\nu t}}$$

(we put 2 because the solution becomes more convenient)
So the final equation is

$$\boxed{\boxed{f'' + 2\eta f' = 0}}$$

To solve this equation, we separate variables:

$$\frac{\mathrm{d}f'}{f'} = -2\eta \mathrm{d}\eta$$

$$\mathrm{d}(\ln f') = -2\eta \mathrm{d}\eta$$

$$\ln f' = -\eta^2 + \hat{\mathcal{C}}_1$$

and

$$f'(\eta) = \mathcal{C}_1 e^{-\eta^2}$$

with $\hat{\mathcal{C}}_1 = \ln \mathcal{C}_1$ With a further integration,

$$f(\eta) = \mathcal{C}_1 \int_0^{\eta} e^{-\eta^2} \mathrm{d}\eta + \mathcal{C}_2$$

Boundary conditions are

$$\begin{cases} \eta \to \infty \text{ that is } y \to \infty, t \to 0) & \Rightarrow f(\eta) = 0 \\ \eta \to 0 & \Rightarrow f(0) = 1 \end{cases}$$

Applying the B.C. for $\eta = 0$,

$$f(0) = \mathcal{C}_1 \int_0^0 e^{-\eta^2} \mathrm{d}\eta + \mathcal{C}_2 = 1$$

The integral behaves well and is zero and therefore $\mathcal{C}_2 = 1$.

$$f(\eta \to \infty) = \mathcal{C}_1 \int_0^\infty e^{-\eta^2} \mathrm{d}\eta + 1$$

We know that $\int_0^\infty e^{-\eta^2} \mathrm{d}\eta = \frac{\sqrt{\pi}}{2}$ and then $\mathcal{C}_1 = -\frac{2}{\sqrt{\pi}}$

The velocity is thus

$$\begin{cases} v_x(y,t) = u\left[1 - \frac{2}{\sqrt{\pi}}\int\limits_0^{\eta} e^{-\eta^2}\mathrm{d}\eta\right] \\ \eta = \frac{y}{2\sqrt{\nu t}} \end{cases}$$

Defining the *error function*

$$erf(\eta) := \frac{2}{\sqrt{\pi}}\int\limits_0^{\eta} e^{-\eta^2}\mathrm{d}\eta$$

we finally obtain

$$\boxed{v_x(y,t) = u\left[1 - erf(\eta)\right]}$$