Modelling of Turbulent Flows

Lecture 2.1

April 3, 2020

Potential Flow

Outline

Potential function

If the flow is irrotational in the entire domain, we can describe the flow field by a suitable function which is called potential, ϕ. It is a scalar function (can be defined in 2 or 4 dimensions being an Hamiltonian) and must satisfy the equation $\boxed{\vec{v} = -\vec{\nabla}\phi}$ where the minus sign is by convention. It follows

$$v_x = -\frac{\partial \phi}{\partial x} \quad ; \quad v_y = -\frac{\partial \phi}{\partial y}$$

ϕ has the same meaning of the potential V in electricity.

We can rewrite the continuity equation in the following way:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = \boxed{\nabla^2 \phi = 0}$$

The problem with this equation is that ϕ must be known to find $\vec{v}$ and that pressure must be found by another equation.

Streamfunction

The 2D field can be described also by another scalar function, ψ, called streamfunction and defined as follows:

$$v_x = -\frac{\partial \psi}{\partial y} \quad ; \quad v_y = +\frac{\partial \psi}{\partial x}$$

which automatically satisfies continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

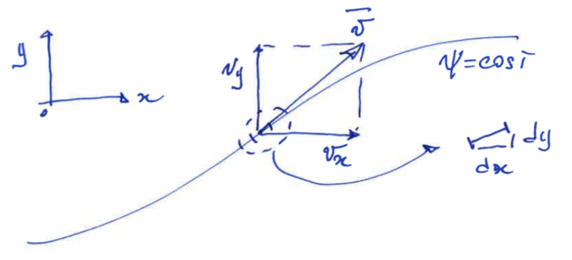
Physical meaning of ψ

The set of all points characterized by the same value of ψ is a streamline. This is everywhere tangent to the velocity vector as shown:

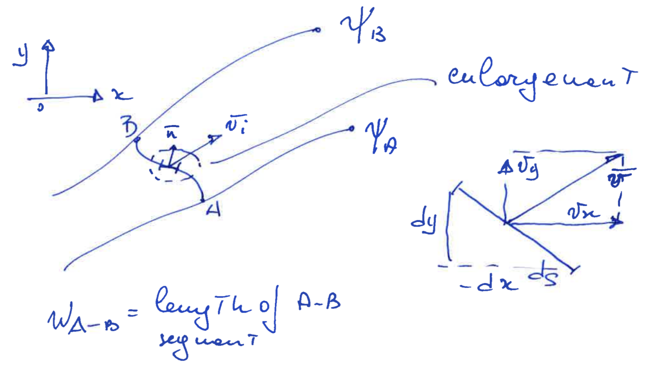
$$\psi = \psi(x, y) \Rightarrow \mathrm{d}\psi = \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial y}\mathrm{d}y = v_y \mathrm{d}x - v_x \mathrm{d}y$$

If we consider a streamline:

$$\psi = cost \Rightarrow \mathrm{d}\psi = 0 \Rightarrow \frac{v_x}{v_y} = \left.\frac{\mathrm{d}x}{\mathrm{d}y}\right|_{\psi=\mathrm{cost}}$$



The **difference** in value **between two streamlines** is the **flowrate** actually flowing between the two streamlines.



$$
\begin{aligned}
\left.\frac{Q}{W}\right|_{A-B} &= \left[\frac{m^3}{ms}\right] \\
&= \int_{A-B} \vec{n}\cdot\vec{v}\,\mathrm{d}S = \int_{A-B}(n_x v_x + n_y v_y)\mathrm{d}S = \\
&= \int_{A-B} v_x(n_x\mathrm{d}S) - \int_{A-B} v_y(n_y\mathrm{d}S) = \\
&= \int_{A-B} v_x\mathrm{d}x - \int_{A-B} v_y\mathrm{d}y = \\
&= \int_{A-B}(-\mathrm{d}\psi) = -(\psi_B - \psi_A) = \psi_A - \psi_B
\end{aligned}
$$

Relations between vorticity, potential and streamfunction

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x \partial y} = 0$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \Rightarrow \boxed{\vec{\omega} = \nabla^2 \psi}$$

These relations are valid in the case of *potential, irrotational flow* (which automatically satisfies the conditions of zero vorticity and continuity). The last unknown quantity in this flow is the pressure, which can be derived by the Navier-Stokes equations.

N-S (dimensionless):

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} P + \frac{1}{Re} \nabla^2 \vec{v}$$

Since we are in the limit $Re \to \infty$,

$$\frac{1}{Re} \nabla^2 \vec{v} \to 0$$

and if the flow is steady ($\frac{\partial \vec{v}}{\partial t} = 0$) the equation becomes (dimensional form):

$$(\vec{v} \cdot \vec{\nabla})\vec{v} + \vec{\nabla} P = 0$$

Bernoulli equation

Now

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla}\left(\frac{1}{2}\vec{v} \cdot \vec{v}\right) - \cancel{\vec{v} \times \vec{\omega}}$$

(zero because of the irrotational flow hypothesis) and

$$\vec{\nabla} P = p + \rho g h$$

so

$$\rho\vec{\nabla}\left(\frac{1}{2}\vec{v} \cdot \vec{v}\right) + \vec{\nabla}(p + \rho g h) = 0$$

$$\vec{\nabla}\left(\frac{1}{2}\rho v^2 + p + \rho g h\right) = 0 \to \frac{1}{2}\rho v^2 + p + \rho g h = \text{cost}$$

which is the **Bernoulli equation**.

This is indeed how Bernoulli derived this equation, not from the energy balance equation. The Bernoulli equation is valid along one streamline. Changing the line makes the constant change too.

Note on the vorticity equation and on boundary conditions

The 2D vorticity equation is a 4th order equation:

$$\text{(A)} \quad \vec{v} \cdot \vec{\nabla}\omega = \frac{1}{Re}\nabla^2\omega$$

$$\text{(B)} \quad \omega = \nabla^2\phi$$

However, equation (B) is 2nd order and describes the flow field. Therefore, usual B.C. cannot be applied:

$\vec{v} \cdot \vec{n} = 0$ no-cross condition;

$\vec{v} \cdot \vec{t} = 0$ free-slip condition

($\vec{t}$ is the tangent versor to the surface; $\vec{n}$ is the normal versor to the surface).
The no-slip condition is thus redundant and not applied.

Paradox

From Greek παράδοξος, παράδοξον (παρά "contrary to" and δόξα "opinion"): an assertion that is essentially self-contradictory, though based on a valid deduction from acceptable premises.

Acceptable premises: If *Re* is very high, only inertial forces are important, since pressure contribution should be balanced by inertial forces and viscus contribution should be negligible. (For instance, hands out of a car window.)

D'Alembert reasoning

D'Alembert wanted to compute the force acting on a circular cylinder immersed into a fluid at high Reynolds number: most of the force must be created by pressure (the blocking effect of the cylinder) and the shear force at the cylinder surface should be negligible.
Therefore, this should be the benchmark example for the Potential Flow theory.

Complex potential

The potential ϕ and the streamfunction ψ both satisfy the Laplace equation and at all points are orthogonal to each other, since:

$$\text{Cauchy-Riemann:} \begin{cases} v_x & = -\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x} \\ v_y & = \frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial y} \end{cases}$$

These relationships, known as **Cauchy-Riemann** relations, must be satisfied by real and imaginary parts of all **analytic functions** $w(z)$ of the complex variable $z = x + iy$.
The function $w(z)$ is the **complex potential**, defined as:

$$w(z) :- \phi(x, y) + i\psi(x, y)$$

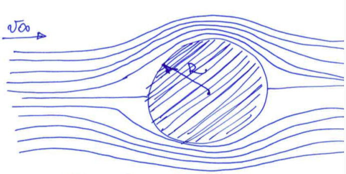
Velocity components can be derived by:

$$\frac{\mathrm{d}w(z)}{z} = -v_x(x, y) + iv_y(x, y)$$

in which $\frac{\mathrm{d}w(z)}{z}$ is the **complex velocity**.

Example

This flow field, describing the streamlines around a circular cylinder, can be described by the complex potential



$$w(z) = v_\infty \left[z + \frac{R^2}{z} \right]$$

Rewriting the complex potential, we can obtain in a straightforward way the potential and the streamfunction:

$$w(z) = \underbrace{v_\infty x \left[1 + \frac{R^2}{x^2 + y^2} \right]}_{\phi(x,y)} + i \underbrace{v_\infty y \left[1 - \frac{R^2}{x^2 + y^2} \right]}_{\psi(x,y)}$$

Using polar coordinates:

$$\phi(r,\theta) = v_\infty \left[r + \frac{R^2}{r} \right] \cos\theta$$
$$\psi(r,\theta) = v_\infty \left[r - \frac{R^2}{r} \right] \sin\theta$$

from which the velocity is easy to obtain:

$$v_r(r,\theta) = -\frac{\partial \phi}{\partial r} = -v_\infty \left[r - \frac{R^2}{r^2} \right] \cos\theta$$
$$v_\theta(r,\theta) = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = v_\infty \left[1 + \frac{R^2}{r^2} \right] \sin\theta$$

We observe that at the cylinder surface $r = R$:

$v_r = 0 \Rightarrow$ No cross condition

$v_\theta = 0$ at $\theta = 0$ and $\theta = \pi$: stagnation points

The flow is perfectly symmetric.

To obtain the pressure we must use the Bernoulli equation without gravity (here p_0 is a reference pressure):

$$
\begin{aligned}
p &= p_0 - \frac{1}{2}\rho v^2 \\
&= p_0 - \frac{1}{2}\rho \left[v_r^2 + v_\theta^2\right] \\
&= p_0 - \frac{1}{2}\rho v_\infty^2 \left[1 + \left(\frac{R^2}{r^2}\right)^2 + r\frac{R^2}{r^2}(\sin^2\theta - \cos^2\theta)\right] \\
&= p_0 - \frac{1}{2}\rho v_\infty^2 \left[\left(1 - \frac{R^2}{r^2}\right)^2 + 4\frac{R^2}{r^2}\sin^2\theta\right] \quad \text{A}
\end{aligned}
$$

Results are:

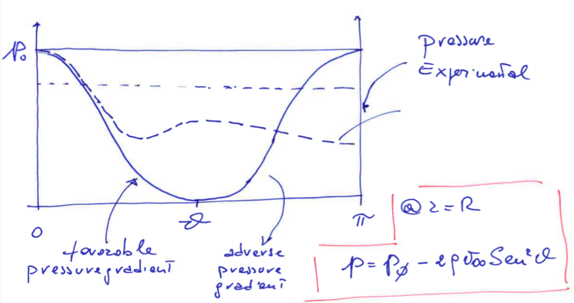
No friction drag (expected)

No form drag (**unexpected!**)

From equation (A) we have that for both the stagnation points

$$p = p_0$$

The predicted pressure distribution is



Explanation

Why does the Potential Flow theory fail?
It is indeed the source of the Paradox: We believe our premises were acceptable because the Reynolds number of the flow was very high; in fact, our premises are *not* acceptable, because precisely in the region where we want to compute the force the local Reynolds number is *small*! Locally, viscous forces become comparable with inertial forces.
Consequence Our model is not just slightly off: it is totally wrong! (when it rains, it pours)