If we are far from the body and the fluid is irrotational (i.e $\omega = 0$) in that region, Kelvin's theorem states that the flow field is irrotational everywhere.

And if the flow is irrotational in the entire domain, we can describe the flow field by a suitable function which is called potential

POTENTIAL FUNCTION, φ

~~The~~ which is a scalar function (can be defined in 2 or 3D being an Hamiltonian).
The potential function must satisfy the equation

$$\boxed{\bar{v} = -\bar{\nabla}\varphi}$$

"$-$" is by convention

$$v_x = -\frac{\partial \varphi}{\partial x} \qquad v_y = -\frac{\partial \varphi}{\partial y}$$

Then, we can rewrite the continuity equation in the following way

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = \nabla^2 \varphi = 0$$

The problem with this equation is that φ must be known to find $\vec{v}$ and that pressure must be found by another equation.

The 2D field can be described also by another scalar function.

THE STREAMFUNCTION ψ

defined as follows

$$\boxed{v_x = -\frac{\partial \psi}{\partial y} \quad ; \quad v_y = +\frac{\partial \psi}{\partial x}}$$

which automatically satisfies continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial \psi}{\partial x \partial y} = 0$$

Physical meaning of φ and ψ

φ has the same meaning of the potential V in electricity.

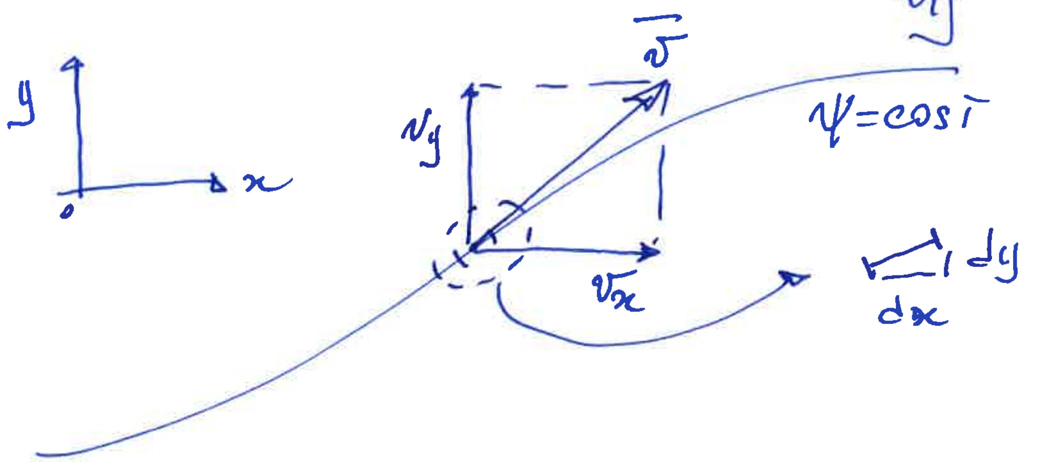
ψ . The set of all points characterized by the same value of ψ is a STREAMLINE. This is everywhere tangent to the velocity vector as shown:

$$\psi = \psi(x, y) \Rightarrow d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy =$$

$$= v_y dx - v_x dy$$

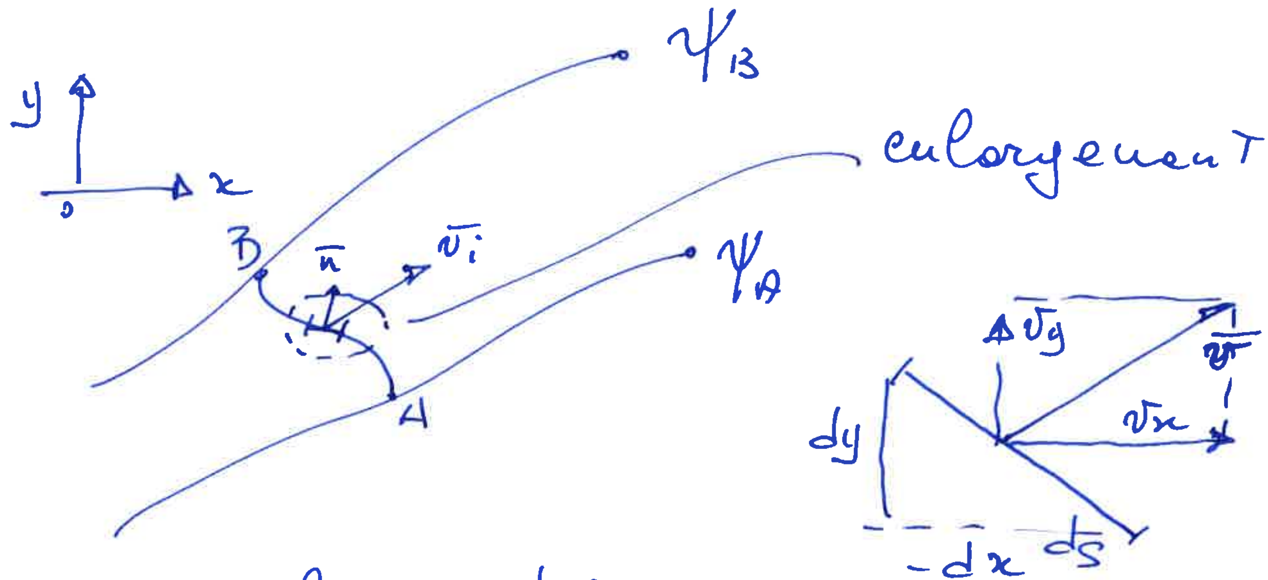
If we consider a streamline ($\psi = \text{cost}$)

$$\psi = \text{cost} \Rightarrow d\psi = 0 \Rightarrow \frac{v_x}{v_y} = \left.\frac{dx}{dy}\right|_{\psi = \text{cost}}$$



This is the local slope of the streamline

The difference in value between two streamlines is the flowrate actually flowing between the two streamlines.



$$w_{A-B} = \text{length of } A\text{-}B \text{ segment}$$

$$\left.\frac{Q}{w}\right|_{A-B} = \int_{A-B} \bar{n}\cdot\bar{v}\, ds = \int_{A-B} (n_x v_x + n_y v_y)\, ds =$$

$$\left[\frac{m^3}{m\, s}\right]$$

$$= \int_{A-B} \overbrace{(n_x ds)}^{dx} v_x - \int_{A-B} \overbrace{(n_y ds)}^{dy} v_y =$$

$$= \int_{A-B} (v_x dy - v_y dx) = \int_{A-B} (-d\psi) =$$

$$= -(\psi_B - \psi_A) = \psi_A - \psi_B$$

Relations between vorticity ω, potential φ, and streamfunction ψ.

$$\omega_z \triangleq \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y \partial x} = 0$$

$$\omega_z \triangleq \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \Rightarrow \boxed{\bar{\omega} = \bar{\nabla}^2 \psi}$$

Poisson Equation

These relations are valid in the case of potential, irrotational flow (which automatically satisfies the conditions of zero vorticity and continuity). The last unknown in this flow is pressure, which can be derived by the Navier-Stokes equations.

N/S (dimensionless) : $\dfrac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{v} = -\bar{\nabla}P + \dfrac{1}{Re}\nabla^2 \bar{v}$

Since we are in the limit $Re \to \infty$ the equation is written as:

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{v} + \bar{\nabla}P = \cancel{\frac{1}{Re}\nabla^2 \bar{v}} \to 0$$

and if the flow is steady:

$$\rho\left(\bar{v}\cdot\bar{\nabla}\right)\bar{v} + \bar{\nabla}P = 0 \quad \Longleftarrow \quad \text{in Dimensional form}$$

$$P = p + \rho g h$$

$$\bar{\nabla}\left(\tfrac{1}{2}\,\bar{v}\cdot\bar{v}\right) - \underbrace{\cancel{\bar{v}\times\bar{\omega}}}_{0}$$

irrotational flow

$$\rho\,\bar{\nabla}\left(\tfrac{1}{2}\,\bar{v}\cdot\bar{v}\right) + \bar{\nabla}\left(p + \rho g h\right) = 0$$

$$\boxed{\cancel{\rho}\,\bar{\nabla}\left(\tfrac{1}{2}\rho v^2 + p + \rho g h\right) = \text{cost}}$$

Bernoulli Equation

This is indeed how Bernoulli derived this equation, not from the energy balance equation ~~energy was not~~

The Bernoulli equation is valid along one streamline. Changing the line the constant changes.

Note on the vorticity equation and on Boundary conditions

The 2D vorticity equation is 4° order equation

$$\text{(A)} \quad \bar{v} \cdot \bar{\nabla} \omega = \frac{1}{Re} \nabla^2 \omega \qquad \text{(B)} \quad [\omega = \nabla^2 \varphi]$$

However, eq. (B) is 2° order and describes the flowfield.

Therefore, usual B.C. cannot be applied

$\bar{v} \cdot \bar{n} = 0$ no-cross condition

$\bar{v} \cdot \bar{t} = 0$ no-slip condition

$\bar{t}$ = Tangent versor to the surface

$\bar{n}$ = normal " " " "

The no-slip condition is thus redundant and not applied

Vorlesung 7.

D'Alembert Paradox

Paradox: An assertion that is essentially self-contradicting though based on a valid deduction from acceptable premises

Acceptable premises: if Re is very high, only inertial forces are important: pressure contribution should be balanced by inertial forces and viscous contribution should be negligible ⊛

In this way had reasoned D'Alembert: he wanted to compute the force acting on a circular cylinder immersed into a fluid at high Reynolds number.

⊛ Example of hands sticking out from the front windows of a car.

Most of the force must be created by pressure (the blocking effect of the cylinder) and the shear force at the cylinder surface should be negligible.

Therefore, this should be the benchmark example for the potential flow theory.

The potential φ and the streamfunction ψ both satisfy the Laplace equation and at all points have orthogonal tangent lines, as we observe from:

Cauchy-Riemann

$$\begin{cases} v_x = -\dfrac{\partial \psi}{\partial y} = -\dfrac{\partial \varphi}{\partial x} \\[2ex] v_y = \dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \varphi}{\partial y} \end{cases}$$

These relationships ~~are~~, known as Cauchy-Riemann relations, must be satisfied by real and imaginary parts of all analytic functions $\boxed{w(z)}$ of the complex variable $\boxed{z = x + iy}$

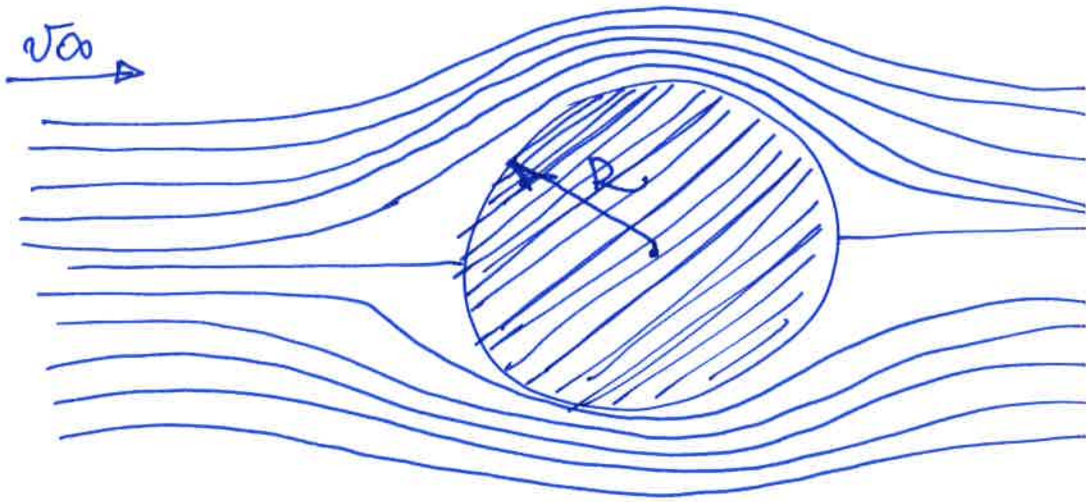
The function $w(z)$ is the COMPLEX POTENTIAL and in general is defined as:

$$w(z) = \varphi(x,y) + i\,\psi(x,y)$$

Velocity components can be derived by:

$$\frac{d\,w(z)}{dz} = -v_x(x,y) + i\,v_y(x,y)$$

in which dw/dz is the complex velocity.



Streamlines around a circular cylinder

This flow field can be described by the complex potential

$$w(z) = v_\infty \left[z + \frac{R^2}{z} \right]$$

If we write the complex potential as:

$$w_z(z) = v_\infty x\left[1+\frac{R^2}{x^2+y^2}\right] + i v_\infty y\left[1-\frac{R^2}{x^2+y^2}\right]$$

We obtain in a straightforward manner potential and streamfunction:

$$\psi(x,y) = v_\infty y\left[1-\frac{R^2}{x^2+y^2}\right]$$

$$\varphi(x,y) = v_\infty x\left[1+\frac{R^2}{x^2+y^2}\right]$$

For the cylinder problem, it is better to use polar coordinates and we obtain

$$\varphi(r,\vartheta) = v_\infty\left[r+\frac{R^2}{r}\right]\cos\vartheta$$

$$\psi(r,\vartheta) = v_\infty\left[r-\frac{R^2}{r}\right]\sin\vartheta$$

From which the velocity is easy to obtain

$$v_r(r,\vartheta) = -\frac{\partial\varphi}{\partial r} = -v_\infty\left[1-\frac{R^2}{r^2}\right]\cos\vartheta$$

$$v_\vartheta(r,\vartheta) = -\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta} = v_\infty\left[1+\frac{R^2}{r^2}\right]\sin\vartheta$$

We observe that at the cylinder surface,

$r = R$

$v_r = 0$ → No cross condition

$v_\theta = 0$ @ $\theta = 0$ and $\theta = \pi$ || Stagnation Points

The flow is perfectly symmetrical.

To obtain the pressure we must use the Bernoulli equation [no gravity]

$$p = p_0 - \frac{1}{2}\rho v^2 =$$

|| P_0 is a reference pressure

$$= p_0 - \frac{1}{2}\rho\left[v_r^2 + v_\theta^2\right] =$$

$$= P_0 - \frac{1}{2}\rho\, v_\infty^2\left[1 + \left(\frac{R^2}{r^2}\right)^2 + 2\frac{R^2}{r^2}\left(\sin^2\theta - \cos^2\theta\right)\right] =$$

$$= p_0 - \frac{1}{2}\rho\, v_\infty^2\left[\left(1 - \frac{R^2}{r^2}\right)^2 + 4\frac{R^2}{r^2}\sin^2\theta\right] \quad \text{(A)}$$

Results are :

- No FRICTION DRAG → Expected
- No FORM DRAG → UNEXPECTED

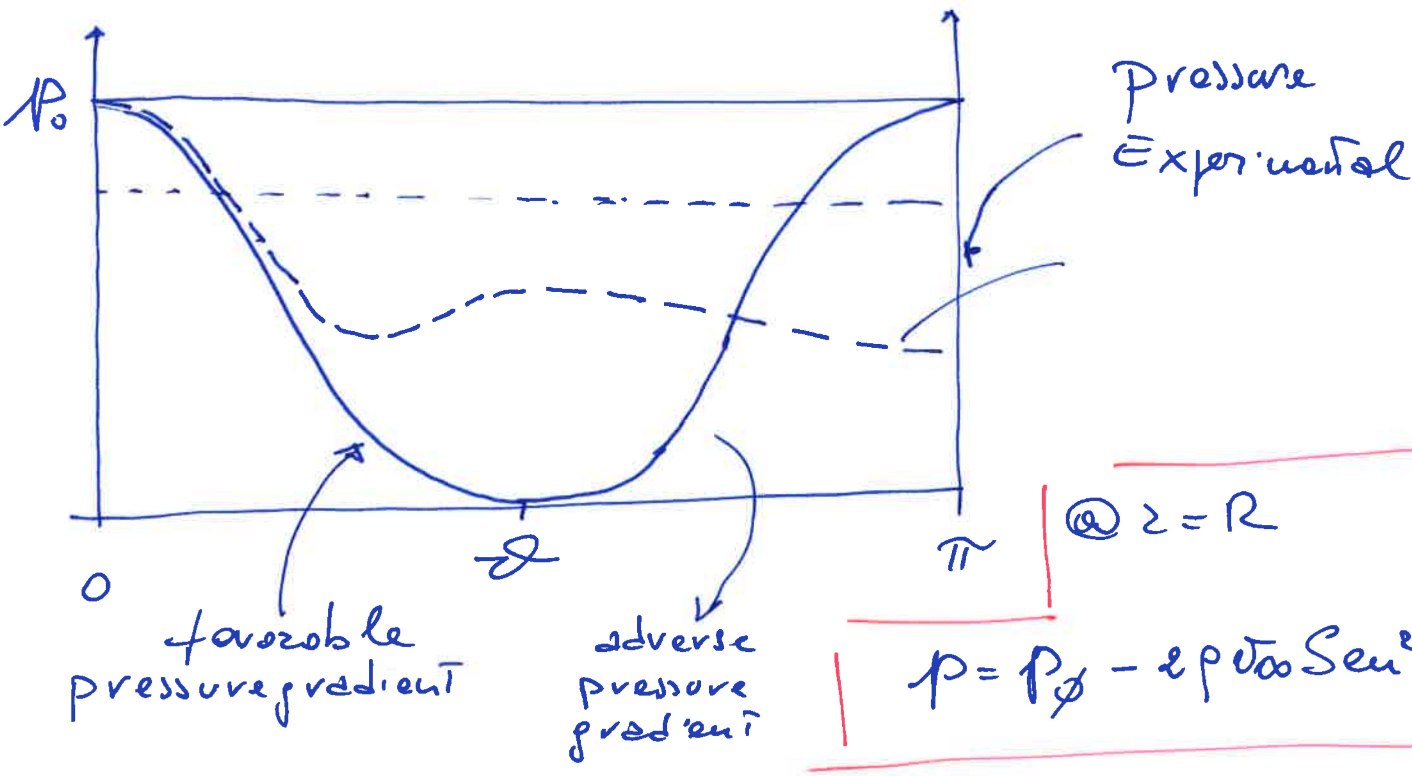
From Eq (A) we have that for the fore stagnation point [$r = R$; $\vartheta = 0$]

$$\boxed{p = p_\phi}$$

And for the aft stagnation point [$r = R$; $\vartheta = \pi$]

$$\boxed{p = p_\phi}$$

The predicted pressure distribution is



@ $r = R$

$$\boxed{p = p_\phi - 2\rho v_\infty \sin^2\vartheta}$$

Why Potential flow Theory does not work?

It is indeed The source of The Paradox:

We BELIEVE our premises were ACCEPTABLE because The flow Reynolds number was very high.

In FACT, our premises are NOT Acceptable because precisely in The region were we want To compute The force The local Reynolds number is small! And locally viscous forces become comparable with inertial forces.

Consequence: our model is not just slightly off. IT is TOTally wrong.

[when iT rains, iT pours]