Modelling of Turbulent Flows

Lecture 1.2

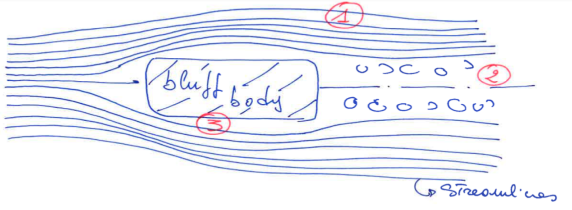
March 24, 2020

Vorticity Dynamics

Outline

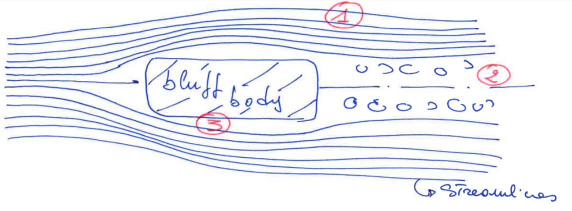
Flow past a Bluff Body

When we consider the flow of a fluid (with density ρ and viscosity μ) past a bluff body we can analyze the flow field in the following way:



We can identify three regions:

Flow past a Bluff Body

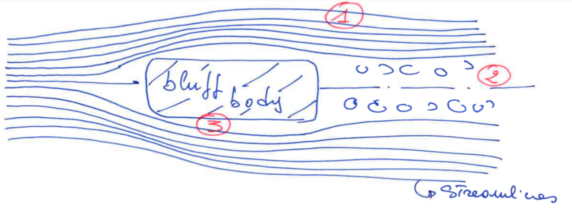


① **Potential flow** Far from the body but with deformation of the streamlines:

Negligible viscous dissipation

Vorticity $= 0$

Flow past a Bluff Body

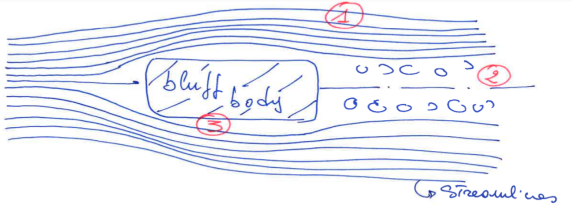


② **Wake region** characterized by vortex stretching:

Negligible viscous dissipation

Non-zero vorticity

Flow past a Bluff Body



③ **Wall region** Boundary layer:

Important viscous dissipation

Non-zero vorticity

The velocity gradient near the wall is important due to the no-slip boundary dissipation due to the viscous stress:

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

Definition of vorticity

$$\vec{\omega} = \mathrm{rot}\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \vec{i} \underbrace{\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right)}_{\omega_x} + \vec{j} \underbrace{\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right)}_{\omega_y} + \vec{k} \underbrace{\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)}_{\omega_z}$$

Vorticity represents the local rotation rate of an elementary parcel of fluid. Since vorticity is defined by the derivatives of the velocity vector, it is also related to the deformation rate.
We can now try to make some examples to understand better the role of vorticity.

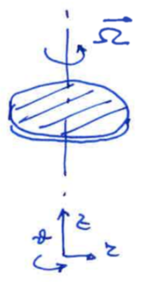
Example 1

A fluid is rotating as if it were a rigid body with angular rotation rate Ω. The velocity field is

$$\vec{u} = \vec{\Omega} \times \vec{r}$$

with $\vec{\Omega} = \Omega_z \hat{k}$ and $\vec{r} = x\vec{i} + y\vec{j}$.

$$\vec{v} = \Omega_z \hat{k} \times (x\hat{i} + j\hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega_z \\ x & y & 0 \end{vmatrix} = \Omega_z \cdot x\hat{j} - \Omega_z \cdot y\hat{i}$$



and $v_z = 0$.

Example 1

The components of vorticity are:

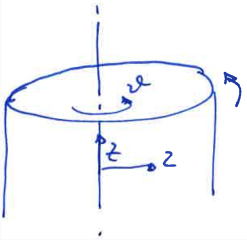
$$\omega_x = \frac{\partial \cancel{v_z}}{\cancel{\partial y}} - \frac{\partial v_y}{\partial z} = -\frac{\partial}{\partial z}(\Omega_z \cdot x) = 0$$
$$\omega_y = \frac{\partial v_x}{\partial z} - \frac{\partial \cancel{v_z}}{\cancel{\partial x}} = \frac{\partial}{\partial z}(\Omega_z \cdot y) = 0$$
$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial}{\partial x}(-\Omega_z \cdot y) - \frac{\partial}{\partial y}(\Omega_z \cdot x) = 2\Omega_z$$

$\Rightarrow \vec{\omega} = \omega_z \hat{k}$ orthogonal to the motion plane.

Example 2

Every fluid particle is moving on a circular path about the z-axis, but with the radial velocity distribution corresponding to the torsional flow.

$v_\theta = \frac{k}{r}$ with $k =$ constant. It is the flow generated in a cylindrical container by the boundary which moves at constant speed. Radial and azimuthal vorticities are null:



$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = 0$$

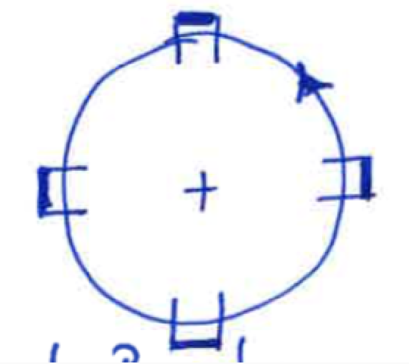
$$\omega_\theta = \frac{\partial v_\theta}{\partial z} - \frac{\partial v_z}{\partial z} = 0$$

while

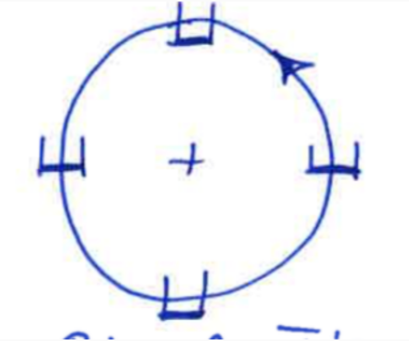
$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_z) - \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}} = \frac{1}{r}\frac{\partial}{\partial r}(r \cdot \frac{k}{r}) = 0$$

Examples 1 & 2

In the limit of  very small:



Rigid body rotation



Circulation without rotation

Example 3

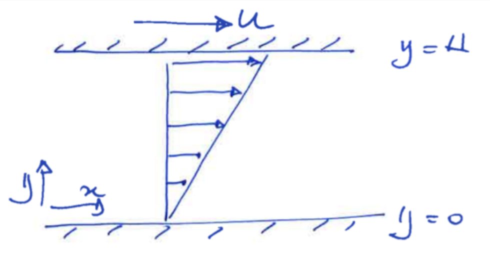
This is the simple shear flow in a Couette device..

$$v_x(y) = \frac{U}{H} y$$

$$\omega_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0$$

$$\omega_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{U}{H}$$

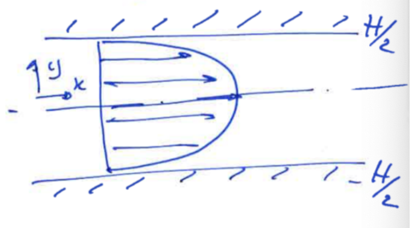


. $\frac{U}{H}$ is, of course, the slope of the velocity profile.

Example 4

Plane Poiseuille flow.

$$v_x(y) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)\left[y^2 - \frac{H^2}{4}\right]$$



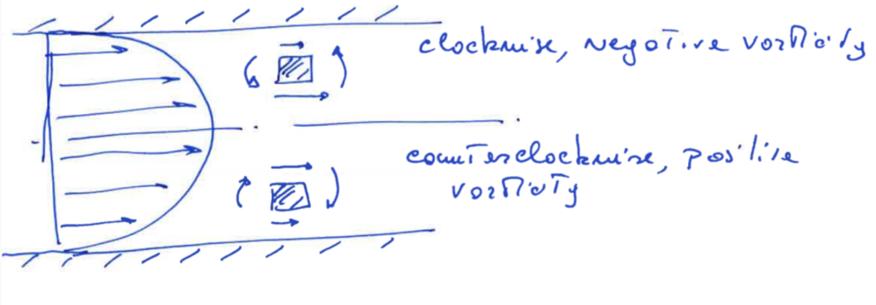
$$\omega_x = \omega_y = 0$$

$$\omega_z = -\frac{\partial v_x}{\partial y} = -\frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)2y$$

$$= -\frac{1}{\mu}\frac{\Delta P}{L}y$$

Maximum vorticity (magnitude) is at both walls. Vorticity is zero in the centerplane.

Example 4


clockwise, negative vorticity
counterclockwise, positive vorticity

Vorticity transport equation

The vorticity transport equation describes the space and time evolution of vorticity. The equation is obtained by applying the curl operation to all terms of the Navier-Stokes equation.

$$\text{rot}\left[\rho\left(\frac{\partial \vec{v}_l}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right)\right] = \text{rot}\left[-\vec{\nabla}P + \mu\nabla^2\vec{v}\right]$$

$$\rho\left(\vec{\nabla}\times\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\times(\vec{v}\cdot\vec{\nabla}\vec{v})\right) = -\vec{\nabla}\times\vec{\nabla}P + \vec{\nabla}\times(\mu\nabla^2\vec{v})$$

$$\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\times\vec{v})}_{\text{vorticity}} + \vec{\nabla}\times(\vec{v}\cdot\vec{\nabla}\vec{v}) = -\frac{\vec{\nabla}\times\vec{\nabla}P}{\rho} + \frac{\mu}{\rho}\vec{\nabla}\times(\nabla^2\vec{v})$$

Some useful vector properties

I. $\forall$ scalar field $A \vec{\nabla} \times \vec{\nabla} A = 0 \Rightarrow \vec{\nabla} \times \vec{\nabla} P = 0$

II. $\vec{\nabla} \times \vec{\nabla} \vec{v} = \nabla^2 \vec{\omega}$

III. $\vec{\nabla} \times (\vec{v} \cdot \vec{\nabla} \vec{v}) = (\vec{v} \cdot \vec{\nabla}) \vec{\omega} - \underbrace{(\vec{\omega} \cdot \vec{\nabla}) \vec{v}}_{=0 \text{ in 2D}}$

because

$$\vec{\nabla} \times (\vec{v} \cdot \vec{\nabla} \vec{v}) = \vec{\nabla} \times \left[\vec{\nabla} \left(\frac{1}{2} \vec{v} \cdot \vec{v} - \vec{v} \times \vec{\omega} \right) \right]$$

$$= \cancel{\vec{\nabla} \times \left[\vec{\nabla} \left(\frac{1}{2} v^2 \right) \right]} - \vec{\nabla} \times (\vec{v} \times \vec{\omega})$$

$$= -\vec{v} (\vec{\nabla} \cdot \vec{\omega}) + \vec{\omega} \underbrace{\cancel{(\vec{\nabla} \cdot \vec{v})}}_{\text{incompr.}} + (\vec{v} \cdot \vec{\nabla}) \vec{\omega} - (\vec{\omega} \cdot \vec{\nabla}) \vec{v}$$

$$= -\vec{v} \cancel{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v})} + (\vec{v} \cdot \vec{\nabla}) \vec{\omega} - (\vec{\omega} \cdot \vec{\nabla}) \vec{v} = (\vec{v} \cdot \vec{\nabla}) \vec{\omega} - \underbrace{(\vec{\omega} \cdot \vec{\nabla}) \vec{v}}_{=0 \text{ in 2D}}$$

Vorticity transport equation

The equation becomes

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\omega}}_{\text{material derivative of } \vec{\omega}} = \underbrace{(\vec{\omega} \cdot \vec{\nabla})\vec{v}}_{\text{vortex stretching; } =0 \text{ in 2D}} + \underbrace{\frac{1}{Re}\nabla^2\vec{\omega}}_{\text{vorticity diffusion}}$$

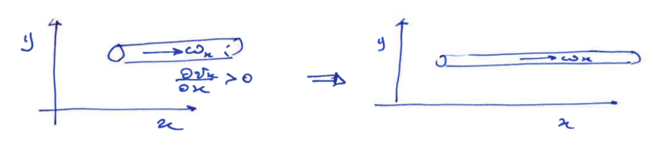
where Re is obtained upon proper non-dimensionalization of the equation.

Vortex stretching term

Considering for simplicity just one component (x), the vortex stretching term is

$$\left[(\vec{\omega} \cdot \vec{\nabla} \vec{v}\right]_x = \underbrace{\omega_x \frac{\partial v_x}{\partial x}}_{\text{vortex stretching part}} + \omega_y \frac{\partial v_x}{\partial y} + \omega_z \frac{\partial v_x}{\partial z}$$

The vortex stretching part acts when a velocity gradient exists in the same direction of vorticity.



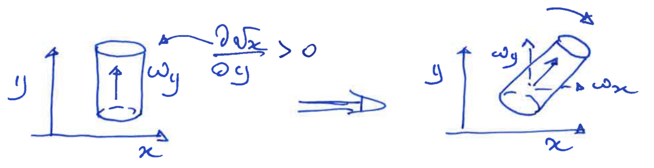
Due to this action, when the fluid parcel is stretched, then, to conserve the angular momentum, there will be a corresponding rotation rate increase and consequently an increase of vorticity (much like the rotation speed of an ice-skate dancer).
N.B. This effect is very important in turbulence because it **helps creating smaller scales**.
This effect is an *auto-amplification* effect: just due to the alignment of velocity gradients and vorticity there is an increase in vorticity.

Vortex stretching term

Considering for simplicity just one component (x), the vortex stretching term is

$$\left[(\vec{\omega} \cdot \vec{\nabla} \vec{v}\right]_x = \omega_x \frac{\partial v_x}{\partial x} + \underbrace{\omega_y \frac{\partial v_x}{\partial y} + \omega_z \frac{\partial v_x}{\partial z}}_{\text{vorticity transfer}}$$

The other two terms, $\omega_y \frac{\partial v_x}{\partial y}$ and $\omega_z \frac{\partial v_x}{\partial z}$, contribute to *rotate* part of the existing vorticity and therefore to **transfer vorticity** from one component to the other.



Baroclinic effect

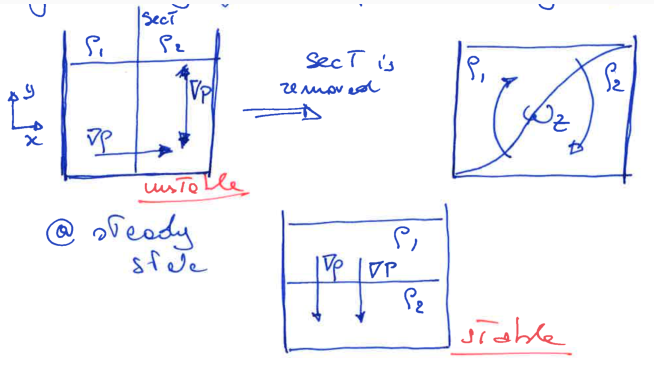
We have considered an incompressible fluid. If we allow difference in density we might have density gradients and the pressure term does not disappear from the vorticity equation, which becomes:

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} = \underbrace{\frac{\vec{\nabla}\rho \times \vec{\nabla}P}{\rho^2}}_{\text{baroclinic term}} + \nu \nabla^2 \vec{\omega} + (\vec{\omega} \cdot \vec{\nabla})\vec{v}$$

Usually, when density is allowed to vary, the density gradient is aligned with the pressure gradient (ocean density, atmospheric density in stable conditions). However, situations may arise when these two gradients are not aligned and vorticity is produced.

Consider the following example:

In a container we have low and high density fluid separated by a sect.



Two-dimensional vorticity equation

In a steady, 2D incompressible flow the vorticity equation is

$$\frac{\partial \vec{\omega}}{\partial t} = 0 \Rightarrow (\vec{v} \cdot \vec{\nabla})\vec{\omega} = \frac{1}{Re}\nabla^2\vec{\omega}$$

Assuming negligible viscous dissipation, we have
$\nu \to 0 \Rightarrow Re \to \infty$ and

$$\boxed{(\vec{v} \cdot \vec{\nabla})\vec{\omega} = 0}$$

In 2D $\vec{\omega} = \omega$ and $\vec{v} \cdot \vec{\nabla}\omega = 0$, which implies that $\vec{v} \perp \vec{\nabla}\omega$.

Kelvin's theorem

Kelvin's theorem states that in an inviscid fluid ($\nu = 0$) the circulation of a material tube is constant. The circulation is

$$\Gamma = \int \omega \mathrm{d}S$$

where $\mathrm{d}S$ is the differential surface of a material tube.

If we are far from the body and the fluid is *irrotational* (i.e. $\omega = 0$) in that region, Kelvin's theorem states that the flow field is irrotational everywhere.