Modelling of Turbulent Flows

Lecture 1.1

March 26, 2021

Conservation equations

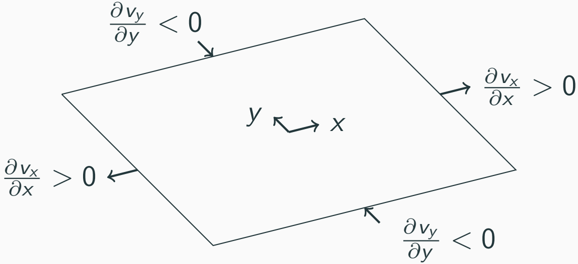
Outline

- Balance Equations
  - Mass Conservation/Continuity
  - Momentum Conservation/ Navier Stokes Equations (for Incompressible Fluids)
  - Different Notations
  - Physical meaning of the terms in the equations
- Adimensionalization of the Balance Equations
  - Why?
  - Adimensionalization is a crucial step in order to look for Approximate Solutions
  - Indeed, we will look for Exact Solutions to Approximate Equations: We will need to know which of the terms of the equations can be discarded because not important

Balance Equations

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

It is possible to visualize the Continuity Equation in two dimensions:



Balance Equations

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=\frac{\partial \mathcal{P}}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j^2} \tag{2}$$

With $\rho =$ Fluid density and $\mu =$ Fluid Viscosity

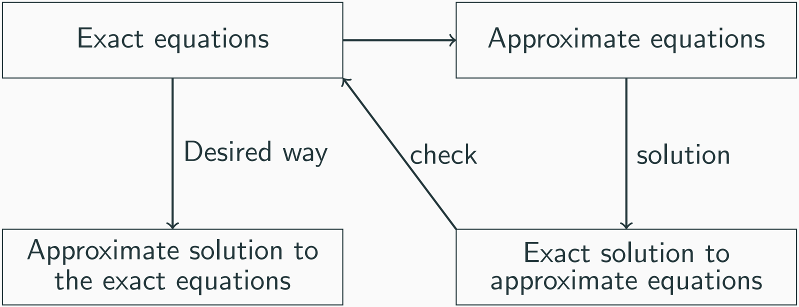
Continuity Equation and Navier-Stokes Equations constitute a set of four (4) differential equations in four unknowns (Pressure $\mathcal{P}$ and the three components of the velocity vector, v_i). In general, this set of equations, although provided by suitable boundary conditions, does not allow for a solution.

Simple cases, in which suitable assumptions lead to the simplifications necessary to solve the equations, are available.

Simplification of the Balance Equations

Our Approach is based on the presumption that **an exact solution of the approximate equations is also an approximate solution of the exact equations**

The rationale of the simplification procedure is depicted in the following scheme.



Simplification of the Balance Equations 2

One nice example is the following:

$$0.01x + y = 0.1 \tag{3}$$

$$x + 101y = 11 \tag{4}$$

If we assume that x and y are of comparable magnitude, we may safely neglect the left term in the first equation and we may obtain $y = 0.1$. Then from the second equation we obtain $x = 0.9$. Now, to check, we plug $x = 0.9$ in the first equation and we obtain that the neglected term is $(0.01x = 0.009)$, which is negligible. So it seems that the method we have used satisfies our requirement. **However, the solution of the full system of equation leads to** $x = -90$ **and** $y = 1$ (L.A. Segel "Simplification and Scaling" SIAM Rev. (1972))

Physical meaning of the terms in the Navier-Stokes Equations

$$\rho \left(\underbrace{\frac{\partial v_i}{\partial t}}_{1} + \underbrace{v_j \frac{\partial v_i}{\partial x_j}}_{2} \right) = \underbrace{\frac{\partial \mathcal{P}}{\partial x_i}}_{3} + \underbrace{\mu \frac{\partial^2 v_i}{\partial x_j^2}}_{4} \tag{5}$$

1 Temporal term (Accumulation term). This term is present if the motion is changing: We cannot do anything about it (we cannot simplify it)

2 Convection term (Non-Linear)

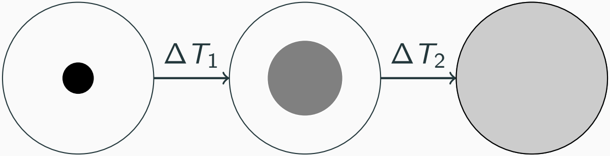
3 Pressure term (In this form it includes the gravity term). This term is the system forcing.

4 Diffusion term

The only option we are left with is to neglect either term 2. or term 4. (convection or diffusion).

Convection and diffusion on momentum in the Navier-Stokes Equations

Diffusion and convection of momentum. Mass transfer analogy:



Molecular diffusion

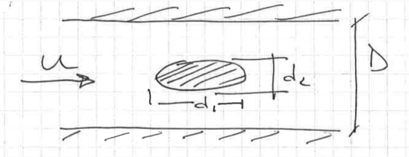
Convection + molecular diffusion

Adimensionalization

The correct adimensionalization is based on the choice of **characteristic variables**.
We have:

- U velocity
- T time [1]
- L space
- Π pressure

$(\nu = \mu/\rho)$



A dimensionless variable is indicated by a tilde: $\widetilde{\bullet}$

[1] Time during which we have appreciable variations. If the flow is pulsed, e.g. heat or engine, then the time scale is fixed.

Nondimensional equations

The solution of a differential equation is not changed by a multiplication constant, so for the uncompressible **continuity equation**

$$\frac{\partial v_i}{\partial x_i} = \frac{\partial v_i/U}{\partial x_i/L} \cdot \frac{U}{L} = \frac{\partial \tilde{v}_i}{\partial \tilde{x}_i} \cdot \frac{U}{L} \Rightarrow \frac{\partial \tilde{v}_i}{\partial \tilde{x}_i} = 0$$

while for *NS equations*

$$\rho \left(\frac{U}{T} \frac{\partial \tilde{v}_i}{\partial \tilde{t}} + \frac{U^2}{L} \tilde{v}_j \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} \right) = -\frac{\Pi}{L} \frac{\partial \tilde{\sigma}}{\partial \tilde{x}_i} + \mu \frac{U}{L^2} \frac{\partial^2 \tilde{v}_i}{\partial \tilde{x}_j^2}$$

If there is <u>no</u> forcing term, $T = \frac{L}{U}$, which can be considered as the flow renewal time.

For sake of simplicity, in the notation we neglect the $\sim$:

$$\rho\left(\frac{U}{T}\frac{\partial v_i}{\partial t}+\frac{U^2}{L}v_j\frac{\partial v_i}{\partial x_j}\right)=-\frac{\Pi}{L}\frac{\partial \mathcal{P}}{\partial x_i}+\mu\frac{U}{L^2}\frac{\partial^2 v_i}{\partial x_j^2}$$

coefficient of inertial term coefficient of viscous term

E.g. if we are on the freeway, going from Wien to Brno, $U = 100km/h$, $L = 100km$, $T = 1h$.

Check dimensionality:

$$\frac{\rho\, U}{T}=\frac{\rho\, U^2}{L}\rightarrow\frac{kg}{m^3}\frac{m^2}{s^2}\frac{1}{m}=\frac{kg}{m^2s^2} \qquad \mu\frac{U}{L^2}\rightarrow\frac{kg}{sm}\frac{m}{s}\frac{1}{m^2}=\frac{kg}{m^2s^2}$$

$$\frac{\Pi}{L}\rightarrow\frac{kg\ m/s^2}{m^2}\frac{1}{m}=\frac{kg}{m^2s^2}$$

Dividing by $\frac{\mu U}{L^2}$

$$\frac{\rho UL}{\mu}\left[\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right]=-\frac{\Pi}{\mu}\frac{L}{U}\frac{\partial \mathcal{P}}{\partial x_i}+\frac{\partial^2 v_i}{\partial x_j^2}$$

Characteristic pressure Π

In general there is no value for Π, it suits the flowfield (e.g. hands out of car windows):

inertial forces dominated flow: $\Pi = \rho U^2$

$$\left(Re := \rho \frac{UL}{\mu} \right)$$

$$\Rightarrow \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial \mathcal{P}}{\partial x_i} = \frac{1}{Re} \frac{\partial^2 v_i}{\partial x_j^2}$$

$Re \to \infty$ **inviscid** fluid; $\mu \to \infty$ (perfect)

viscous forces dominated flow: $\Pi = \frac{\mu U}{L}$

$$\Rightarrow \frac{\rho UL}{\mu} \left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_i} \right] = -\frac{\partial \mathcal{P}}{\partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2}$$

$Re \to 0$ **creeping** flow; $\rho \to 0$