# Fluid Mechanics II

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Donnerstag 10–12 Uhr Lesson 1

- Balance Equations.
  - Mass Conservation / Continuity.
  - Momentum Conservation / Navier–Stokes Eq. [for incompressible fluids]
  - Different notations.
  - Physical meaning of the terms in the equations.

- Adimensionalization of the balance Equations.

Why?

⇒ This is crucial since we are now looking (from now on) to

- $\boxed{\text{Approximate Solutions}}$

- Indeed, we will look for EXACT SOLUTIONS

TO APPROXIMATE EQUATIONS. We will need to know which of the terms of the equations can be discarded because not important.

SUMMARY

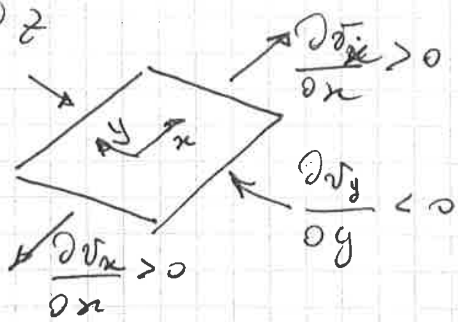
■ Mass Conservation for incompressible fluid

$$\frac{\partial v_i}{\partial x_i} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

2D concept of continuity

eq.

$$\frac{\nabla \bar{v}_k}{\cancel{\partial}} = 0$$




■ Momentum Conservation for Newtonian, incompressible fluid.

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2}$$

$\rho$ = Density ; $\mu$ = Viscosity.

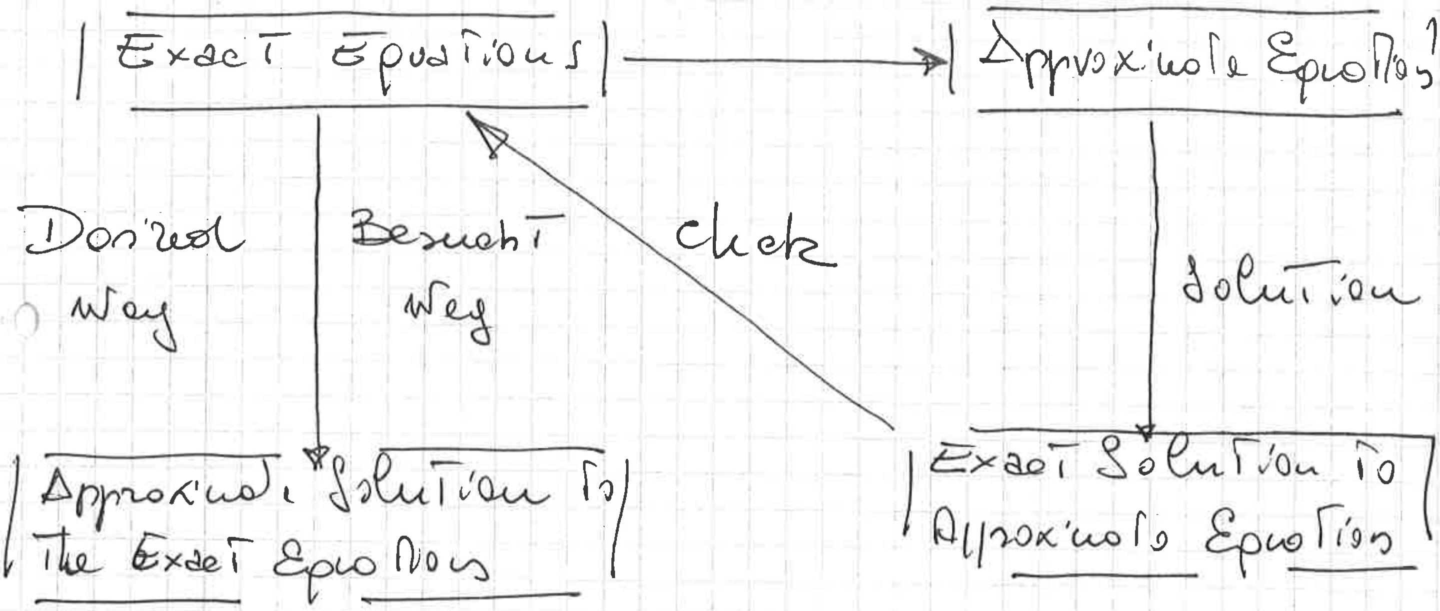
These equations ~~...~~, constitute a set of four differential equations in four unknowns ($p$, $\bar{v}$) which IN GENERAL ~~are not~~ do not allow for a solution.

To solve them, we need to identify simple cases in which is possible to neglect some of the terms.

The RATIONALE of our procedure will be the one depicted in the following scheme.

Our approach is based on the presumption that

"an exact solution of the approximate equations is also an approximate solution to the exact equations"




The presumption needs not to be true in all cases. It is based on some preliminary work. Missing this work, may lead us to wrong conclusions.

Nice example

$$\begin{cases} \underbrace{0{,}01\,x}_{\sim 0} + y = 0{,}1 \\ x + 101\,y = 11 \end{cases}$$

If we assume that $x$ and $y$ are of comparable magnitude we may safely neglect the left term in the first eq. and obtain $y = 0{,}1$

Then, from the $2^{nd}$ eq. $x = 0,9$ -

Now the check:

plug $x = 0,9$ in the $1^{st}$ eq and we obtain that the neglected term $(0,01\, x = 0,009)$ which is negligible -

So it seems that the method satisfies what we require -

HOWEVER, the solution to the full system of equation is $x = -90$, $y = 1$

L. A. Segel "simplification of scaling" SIAM Rev. (1972).

Our mistake was to hypothesize that all unknown had SAME order of magnitude and the value the coefficients were enough to size the term -

———— . ————

Physical meaning of the Terms in the Equation.

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2}$$

(1) (2) (3) (4)

(1) Temporal Term [accumulation Term]

(2) Convection Term [non-linear]

(3) Pressure Term [it includes, in its form, the gravity head]

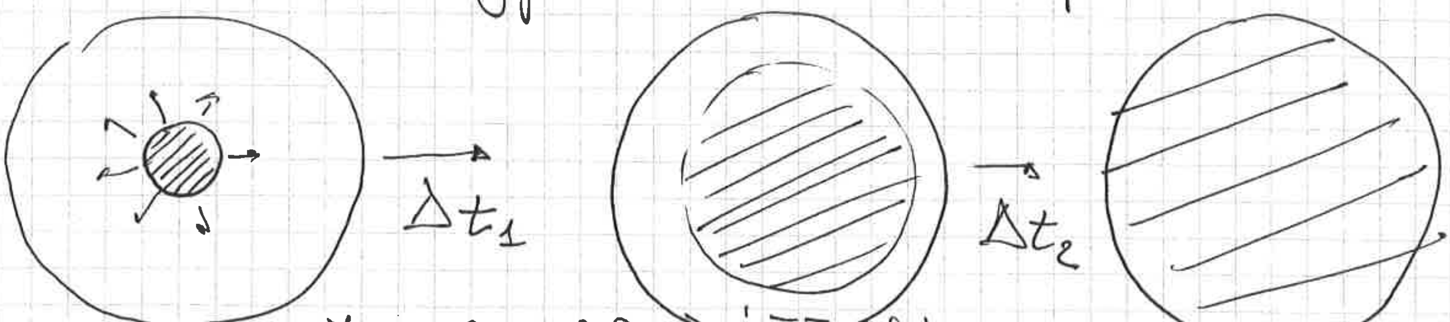
(4) Diffusion Term.

(1) This Term is present if the motion is changing. We cannot do anything about it!

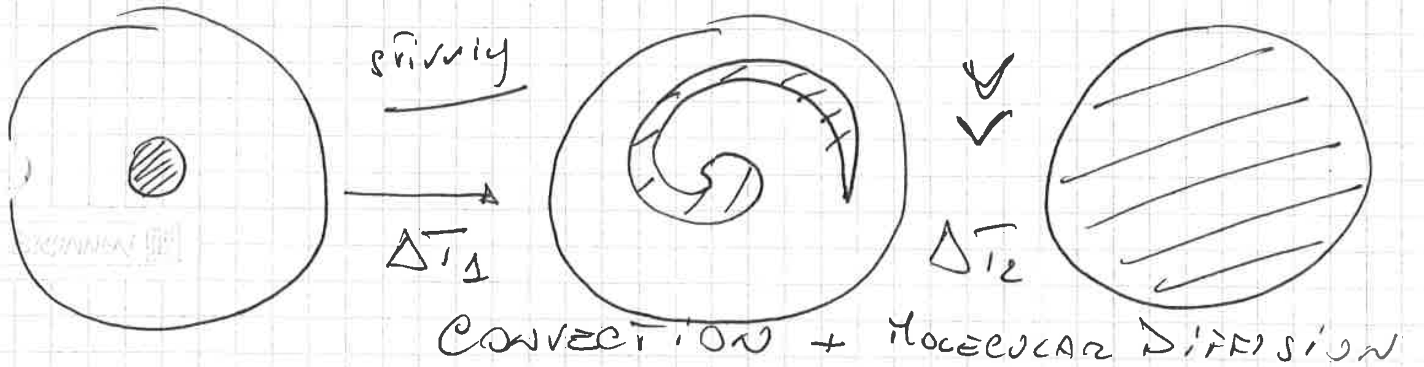
(3) The pressure Term is the system forcing.

(2) & (4) We can choose whether to neglect the (2) and the (4) Term.

From an analogy with mass Transfer:




MOLECULAR DIFFUSION



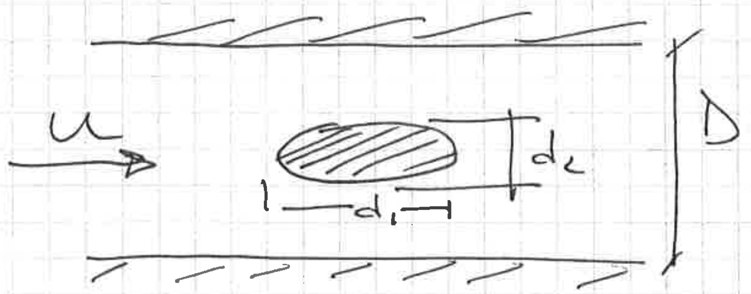

CONVECTION + MOLECULAR DIFFUSION

To choose among diffusion and convection we must ensure that all ~~unknowns~~ variables (dependent and independent) are of the same order.

→ ADIMENSIONALIZATION

The correct adimensionalization is based on the choice of $\boxed{\text{characteristic variables}}$ Freeway example

Suppose we have the following flow:




Fluid physical properties $\rho$ and ~~$\mu$~~ are constant $\nu\left[\mu/\rho\right]$

In our equations we have

| | |
|---|---|
| Velocity | $U$ |
| Time | $T$ |
| Space | $L$ |
| Pressure | $\Pi$ |

$T$ = Time during which we have appreciable variations. If the flow is pulsed, i.e. heart or engine, then the time scale is fixed.

Variable
Dimensionless variable $\tilde{\cdot}$ ⇒ $\tilde{v}_i = \dfrac{v_i}{U}$ etc

Adimensionalization of the continuity equation -

$$\boxed{\frac{\partial v_i}{\partial x_i} \Rightarrow \frac{\partial v_i/U}{\partial x_i/L}\cdot\frac{U}{L} = \frac{\partial \tilde{v}_i}{\partial \tilde{x}_i}\cdot\frac{U}{L} = \frac{\partial \tilde{v}_i}{\partial \tilde{x}_i}}$$

But the solution of a differential equation is not changed by a ~~coeff~~ multiplicative constant

N-S. $\rho\left(\frac{U}{T}\frac{\partial \tilde{v}_i}{\partial \tilde{t}} + \frac{U^2}{L}\tilde{v}_j\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\right) = -\frac{\Pi}{L}\frac{\partial \tilde{\sigma}}{\partial \tilde{x}_i} + \mu\frac{U}{L^2}\frac{\partial^2 \tilde{v}_i}{\partial \tilde{x}_j^2}$

If there is no forcing term, $T = \frac{L}{U}$

which can be considered as the flow renewal time

$\Rightarrow$ If for simplicity of notation we neglect the $\sim$ $\Rightarrow$ $\tilde{\bullet} = \bullet$

(A) $\Rightarrow \rho\left[\underbrace{\frac{U^2}{L}}_{\text{coeff. of iner. term}}\frac{\partial v_i}{\partial t} + \frac{U^2}{L}v_j\frac{\partial v_i}{\partial x_j}\right] = -\frac{\Pi}{L}\frac{\partial \sigma}{\partial x_i} + \underbrace{\frac{\mu U}{L^2}}_{\text{coeff. of viscous term}}\frac{\partial^2 v_i}{\partial x_j^2}$

but now all variables are dimensionless.

It is like we are on the freeway, we have to go ~~from~~ to Brno $U = 100\,\mathrm{Km/hr}$ ; $L = 100\,\mathrm{Km}$ ; $T = \frac{L}{U} = 1\mathrm{h}$

dividing (A) by $\mu\, U/L^2$

(B) $\rho\frac{UL}{\mu}\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right] = -\frac{\Pi L}{\mu U}\frac{\partial \sigma}{\partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2}$

$= \mathrm{Kg}\frac{\mathrm{m}}{\mathrm{s}^2}$

$\rho\frac{U^2}{L}$ dimensions $\frac{\mathrm{Kg}}{\mathrm{m}^3}\frac{\mathrm{m}^2}{\mathrm{s}^2}\cdot\frac{1}{\mathrm{m}} = \frac{\mathrm{Kg}}{\mathrm{s}^2}$ $\left[\begin{array}{l}\text{Force}\\\text{x unit}\\\text{area}\end{array}\right]$

$\mu\frac{U}{L^2} = \cancel{\frac{\mathrm{Pa}\cdot\mathrm{s}}{1}\cdot\frac{\mathrm{m}}{\mathrm{s}}\cdot\frac{1}{\mathrm{m}^2} = \frac{\mathrm{N}}{\mathrm{s}\,\mathrm{m}^2}\cdot\frac{\mathrm{m}}{\mathrm{s}}\cdot\frac{1}{\mathrm{m}^2} = \frac{\mathrm{Kg}\,\mathrm{m}}{\mathrm{s}^2}\cdot\frac{1}{\mathrm{m}^2}\cdot\frac{\mathrm{m}}{\mathrm{s}}\cdot\frac{1}{\mathrm{m}^2}} =$

$\frac{\mathrm{N}}{\mathrm{m}^2}$ $\frac{\mathrm{Kg}}{\mathrm{m}\,\mathrm{s}}\cdot\frac{\mathrm{m}}{\mathrm{s}}\cdot\frac{1}{\mathrm{m}^2} = \frac{\mathrm{Kg}}{\mathrm{s}^2}$ OK

Characteristic Pressure $\Pi$

In general, there is no value for $\Pi$: it suits the flowfield.

Example of the hands while driving.

- ~~Flo~~ Inertial Forces dominated Flow

$$\Pi = \rho U^2$$

- Viscous Forces dominated Flow

$$\Pi = \mu \frac{U}{L}$$

Therefore, the N-S equation becomes.

A) Viscosity effect dominated flow.

$$\rho \frac{UL}{\mu} \left[ \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] = - \frac{\partial \sigma}{\partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2}$$

B) Inertia dominated flow

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial \sigma}{\partial x_i} = \frac{1}{\frac{\rho U L}{\mu}} \frac{\partial^2 v_i}{\partial x_j^2}$$

and $\rho \frac{UL}{\mu} = Re$,

B) $Re \to \infty$ Inviscid fluid; $\mu \to 0$ (Perfect)

A) $Re \to 0$ Fluid with $\rho \to 0$

creeping flow