Vorlesung 7.

D'Alembert Paradox

Paradox: An assertion that is essentially self-contradicting though based on a valid deduction from acceptable premises

- Acceptable premises: if Re is very high, only inertial forces are important: pressure contribution should be balanced by inertial forces and viscous contribution should be negligible (*)

In this way had reasoned D'Alembert: he wanted to compute the force acting on a circular cylinder immersed into a fluid at high Reynolds number.

(*) Example of hands sticking out from the forewindows of a car.

Most of the force must be created by pressure (the blocking effect of the cylinder) and the shear force on the cylinder surface should be negligible.
Therefore, this should be the benchmark example for the Potential flow Theory.

The potential φ and the streamfunction ψ both satisfy the Laplace equation and at all points have orthogonal tangent lines, as we observe from:

Cauchy-Riemann

$$\begin{cases} v_x = -\dfrac{\partial \psi}{\partial y} = -\dfrac{\partial \varphi}{\partial x} \\[2ex] v_y = \dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \varphi}{\partial y} \end{cases}$$

These relationships ~~are~~, known as Cauchy-Riemann relations, must be satisfied by real and imaginary parts of ~~each~~ all analytic functions $\boxed{w(z)}$ of the complex variable ~~z = x + iy~~ $\boxed{z = x + iy}$

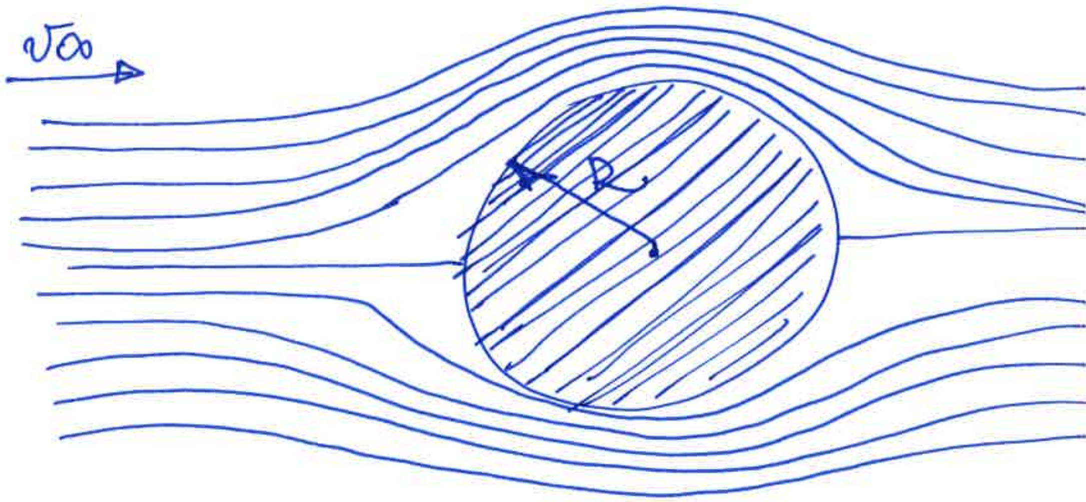
The function $w(z)$ is the COMPLEX POTENTIAL and in general is defined as:

$$w(z) = \varphi(x,y) + i\,\psi(x,y)$$

Velocity components can be derived by:

$$\frac{d\,w(z)}{dz} = -\,v_x(x,y) + i\,v_y(x,y)$$

in which dw/dz is the complex velocity.



Streamlines around a circular cylinder

This flow field can be described by the complex potential

$$w(z) = v_\infty \left[z + \frac{R^2}{z} \right]$$

If we write the complex potential as:

$$w_z(z) = v_\infty x \left[1 + \frac{R^2}{x^2+y^2}\right] + i v_\infty y \left[1 - \frac{R^2}{x^2+y^2}\right]$$

We obtain in a straightforward manner potential and streamfunction:

$$\psi(x,y) = v_\infty y \left[1 - \frac{R^2}{x^2+y^2}\right]$$

$$\varphi(x,y) = v_\infty x \left[1 + \frac{R^2}{x^2+y^2}\right]$$

For the cylinder problem, it is better to use polar coordinates and we obtain

$$\varphi(r,\vartheta) = v_\infty \left[r + \frac{R^2}{r}\right] \cos\vartheta$$

$$\psi(r,\vartheta) = v_\infty \left[r - \frac{R^2}{r}\right] \sin\vartheta$$

From which the velocity is easy to obtain

$$v_r(r,\vartheta) = -\frac{\partial\varphi}{\partial r} = -v_\infty \left[1 - \frac{R^2}{r^2}\right] \cos\vartheta$$

$$v_\vartheta(r,\vartheta) = -\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta} = v_\infty \left[1 + \frac{R^2}{r^2}\right] \sin\vartheta$$

We observe that at the cylinder surface,

$r = R$

$v_r = 0$ $\longrightarrow$ No cross condition

$v_\vartheta = 0$ @ $\vartheta = 0$ and $\vartheta = \pi$ || Stagnation Points

The flow is perfectly symmetrical.

To obtain the pressure we must use the Bernoulli equation [no gravity]

$$p = p_0 - \frac{1}{2}\rho v^2 =$$

|| p_0 is a reference pressure

$$= p_0 - \frac{1}{2}\rho\left[v_r^2 + v_\vartheta^2\right] =$$

$$= p_0 - \frac{1}{2}\rho\, v_\infty^2\left[1 + \left(\frac{R^2}{r^2}\right)^2 + 2\frac{R^2}{r^2}\left(\sin^2\vartheta - \cos^2\vartheta\right)\right] =$$

$$= p_0 - \frac{1}{2}\rho\, v_\infty^2\left[\left(1 - \frac{R^2}{r^2}\right)^2 + 4\frac{R^2}{r^2}\sin^2\vartheta\right] \qquad (A)$$

Results are :

- No Friction Drag → Expected
- No Form Drag → Unexpected

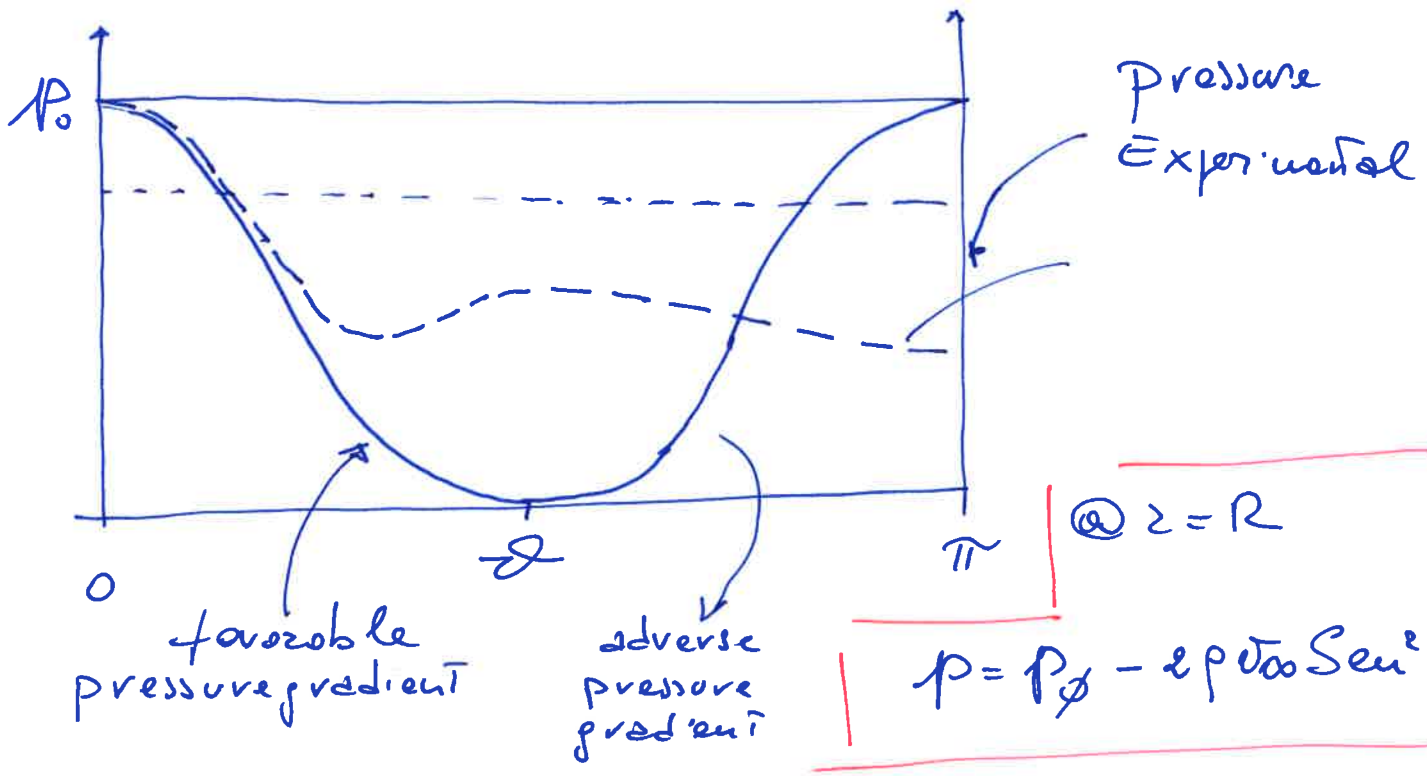
From Eq (A) we have that for the fore stagnation point [$r = R$; $\vartheta = 0$]

$$\boxed{p = p_\varnothing}$$

And for the aft stagnation point [$r = R$; $\vartheta = \pi$]

$$\boxed{p = p_\varnothing}$$

The predicted pressure distribution is



$$\boxed{@\ r = R \qquad p = p_\varnothing - 2 \rho v_\infty^2 \sin^2\vartheta}$$

Why Potential flow Theory does not work?

It is indeed the source of the Paradox:

We BELIEVE our premises were ACCEPTABLE because the flow Reynolds number was very high.

In FACT, our premises are NOT acceptable because precisely in the region were we want to compute the force the local Reynolds number is small! And locally viscous forces become comparable with inertial forces.

Consequence: our model is not just slightly off. It is totally wrong. [when it rains, it pours]