Vorticity Transport Equation

The vorticity transport equation describes the space and time evolution of vorticity. The equation is obtained by applying the curl operation to all terms of the Navier-Stokes equation.

$$(1) \quad \mathrm{rot}\left[\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}P + \mu\vec{\nabla}^2\vec{v}\right]$$

$$(2) \quad \rho\left(\vec{\nabla}\times\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\times(\vec{v}\cdot\vec{\nabla}\vec{v})\right) = -\vec{\nabla}\times\vec{\nabla}P + \vec{\nabla}\times(\mu\nabla^2\vec{v})$$

$$(3) \quad \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{v}) + \vec{\nabla}\times(\vec{v}\cdot\vec{\nabla}\vec{v}) = -\frac{\vec{\nabla}\times\vec{\nabla}P}{\rho} + \frac{\mu}{\rho}\vec{\nabla}\times\vec{\nabla}^2\vec{v}$$

This is the vorticity

We can simplify eq. (3) taking advantage of the following vector properties

I) $\forall$ scalar field $A \Rightarrow \overline{\nabla}\times\overline{\nabla}A = 0$

and $\to$ $\overline{\nabla}\times\overline{\nabla}P = 0$

II) $\overline{\nabla}\times\overline{\nabla}\,\overline{v} = \nabla^2\overline{\omega}$

III) $\overline{\nabla}\times(\overline{v}\cdot\overline{\nabla}\,\overline{v}) = \overline{\nabla}\times\left[\overline{\nabla}\left(\underbrace{\tfrac{1}{2}\overline{v}\cdot\overline{v}}_{\frac{1}{2}v^2}\right) - \overline{v}\times\overline{\omega}\right] =$

$= \overline{\nabla}\times\left[\cancel{\overline{\nabla}\left(\tfrac{1}{2}v^2\right)}\right]_{0} - \overline{\nabla}\times(\overline{v}\times\overline{\omega}) =$

$\overline{\nabla}\cdot\overline{\omega} = \overline{\nabla}\cdot(\overline{\nabla}\times\overline{v})$ $= 0 \ \forall$ vector field

div $\overline{v} = 0$ [incomp.]

$= -\overline{v}\,(\cancel{\overline{\nabla}\cdot\overline{\omega}})_{0} + \overline{\omega}\,(\cancel{\overline{\nabla}\cdot\overline{v}}) +$

0 if 2D

$+ (\overline{v}\cdot\overline{\nabla})\overline{\omega} - \cancel{(\overline{\omega}\cdot\overline{\nabla})\overline{v}} =$

$= (\overline{v}\cdot\overline{\nabla})\overline{\omega}$

The vorticity transport equation becomes

$$\boxed{\frac{\partial\overline{\omega}}{\partial t} + (\overline{v}\cdot\overline{\nabla})\overline{\omega} = \frac{1}{Re}\nabla^2\overline{\omega}} \quad 2D$$

$$\boxed{\frac{\partial\overline{\omega}}{\partial t} + (\overline{v}\cdot\overline{\nabla})\overline{\omega} = (\overline{\omega}\cdot\overline{\nabla})\overline{v} + \frac{1}{Re}\nabla^2\overline{\omega}} \quad 3D$$

and Re comes out by making properly dimensionless the equation.

This equation has three terms which indicate:

$$\underbrace{\frac{\partial \bar{\omega}}{\partial t} + (\bar{v} \cdot \bar{\nabla})\bar{\omega}}_{\text{Material Derivative of } \bar{\omega}} = \underbrace{(\bar{\omega} \cdot \bar{\nabla})\bar{v}}_{\text{Vortex Stretching}} + \underbrace{\frac{1}{Re} \nabla^2 \bar{\omega}}_{\text{Vorticity Diffusion}}$$

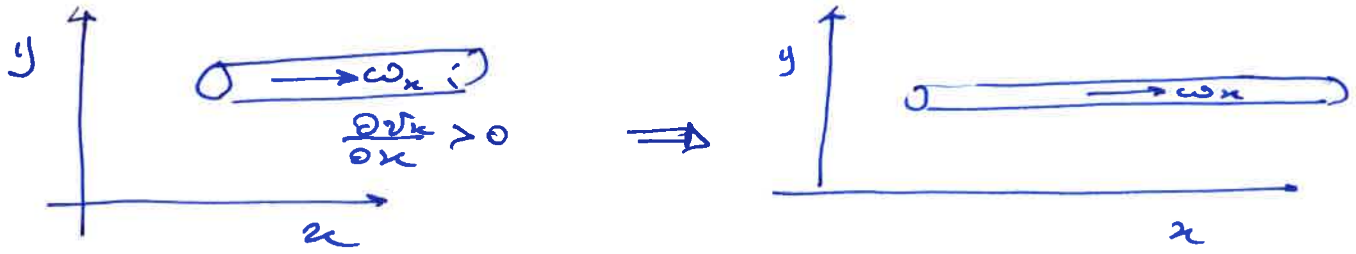
■ Analysis of the Vortex stretching Term

Considering for simplicity just one, x, component, the Vortex Stretching term is

$$x) \quad \left[(\bar{\omega} \cdot \bar{\nabla})\bar{v}\right]_x = \underbrace{\omega_x \frac{\partial v_x}{\partial x}} + \omega_y \frac{\partial v_x}{\partial y} + \omega_z \frac{\partial v_x}{\partial z}$$

→ This is the vortex stretching part of the term.

This part acts when a velocity gradient exists in the same direction of vorticity.

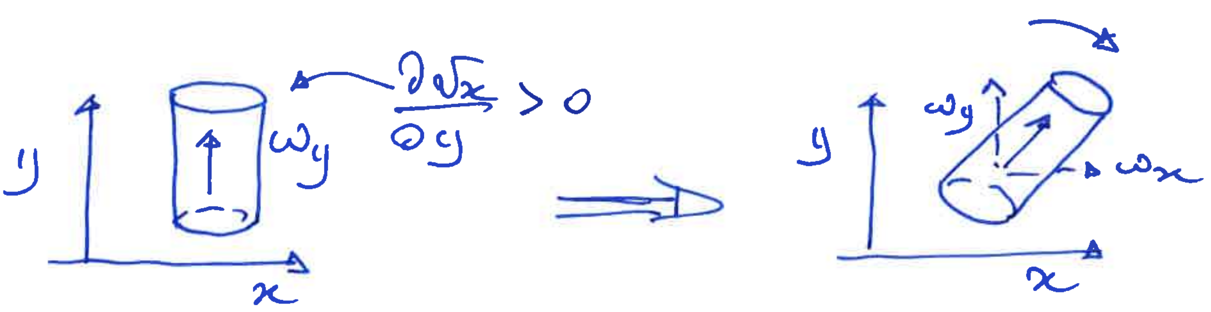


Due to this action, when the fluid parcel is stretched, then, to conserve the angular momentum, there will be a corresponding rotation rate increase and consequently an increase of vorticity. Much like the rotation speed of an ~~sk~~ ice-skate dancer.

<u>N.B.</u> This effect is very important in Turbulence because it helps creating smaller scales.

This effect is an <u>Autoamplification</u> effect. Just due to the alignment of velocity gradients and vorticity there is an increase in vorticity.

The other two terms, $\omega_y \frac{\partial v_x}{\partial y}$ and $\omega_z \frac{\partial v_x}{\partial z}$ contribute to rotate part of the existing vorticity and therefore to <u>transfer vorticity</u> from one component to the other



Note | Baroclinic Effect.

We have considered an incompressible fluid. If we allow difference in density we might have density gradients and the pressure term does not disappear from the vorticity equation which becomes:

$$\frac{\partial \bar{\omega}}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{\omega} = \boxed{\frac{\vec{\nabla}\rho \times \vec{\nabla}P}{\rho^2}} + \nu\nabla^2\bar{\omega} + (\bar{\omega}\cdot\bar{\nabla})\bar{v}$$

Baroclinic Term

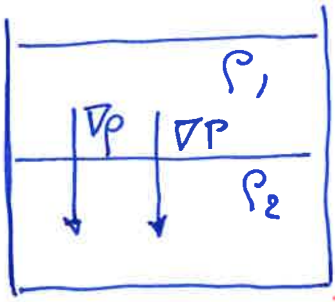
Usually, when density is allowed to vary, the density gradient is aligned with the pressure gradient (ocean density, atmospheric density in stable conditions). However, situations may arise when these two gradients are not aligned and vorticity is produced.

Consider the following example:

In a container we have low and high density fluid separated by a sect.

unstable

@ steady state



stable

Potential Flow and Two-Dimensional Vorticity equation

In a steady, 2D incompressible flow the vorticity equation is

$$\frac{\partial \overline{\omega}}{\partial t} = 0 \Rightarrow \boxed{(\overline{v} \cdot \overline{\nabla})\overline{\omega} = \frac{1}{Re} \nabla^2 \overline{\omega}}$$

Assuming viscous dissipation negligible, we have $\nu = 0 \rightarrow Re \rightarrow \infty$ and

$$\boxed{(\overline{v} \cdot \overline{\nabla}) \cdot \overline{\omega} = 0}$$

In 2D $\overline{\omega} \equiv \omega$

and $\overline{v} \cdot \overline{\nabla} \omega = 0$ which implies that $\overline{v} \perp \overline{\nabla} \omega$

Kelvin's Theorem states that in an inviscid fluid ($\nu = 0$) the circulation of a material tube is constant.

The circulation is $\boxed{\Gamma = \int \omega \, dS}$

where dS is the differential surface of a material tube.

If we are far from the body and the fluid is irrotational (i.e $\omega = 0$) in that region, Kelvin's theorem states that the flow field is irrotational everywhere.

And if the flow is irrotational in the entire domain, we can describe the flow field by a suitable function which is called potential

POTENTIAL FUNCTION, φ

~~The~~ which is a scalar function (can be defined in 2 or 3D being an Hamiltonian).
The potential function must satisfy the equation $\boxed{\bar{v} = -\bar{\nabla}\varphi}$

"−" is by convention

$$v_x = -\frac{\partial \varphi}{\partial x} \qquad v_y = -\frac{\partial \varphi}{\partial y}$$

Then, we can rewrite the continuity equation in the following way

$$\boxed{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = \nabla^2 \varphi = 0}$$

The problem with this equation is that φ must be known to find $\vec{v}$ and that pressure must be found by another equation.

The 2D field can be described also by another scalar function.

THE STREAMFUNCTION ψ

defined as follows

$$\boxed{v_x = -\frac{\partial \psi}{\partial y} \quad ; \quad v_y = +\frac{\partial \psi}{\partial x}}$$

which automatically satisfies continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial \psi}{\partial x \partial y} = 0$$

Physical meaning of φ and ψ

φ has the same meaning of the potential V in electricity.

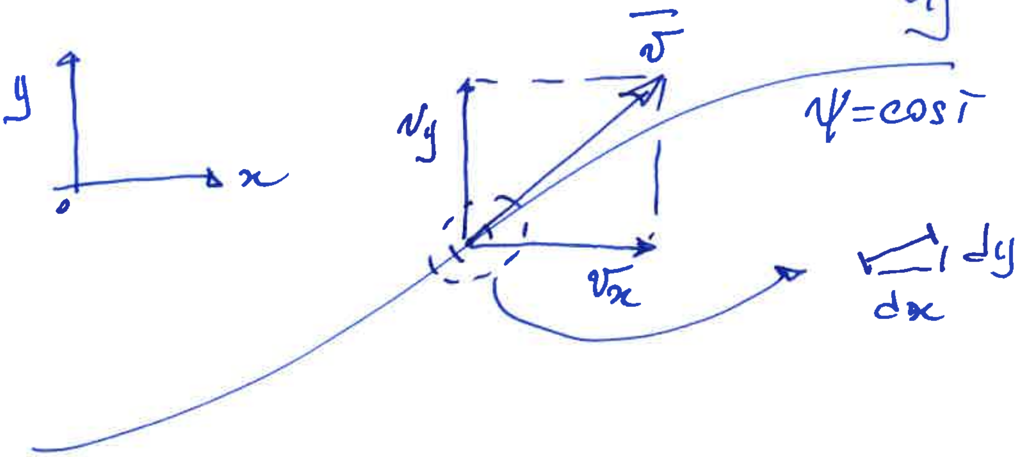
ψ. The set of all points characterized by the same value of ψ is a STREAMLINE. This is everywhere tangent to the velocity vector as shown:

$$\psi = \psi(x, y) \Rightarrow d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy =$$

$$= v_y dx - v_x dy$$

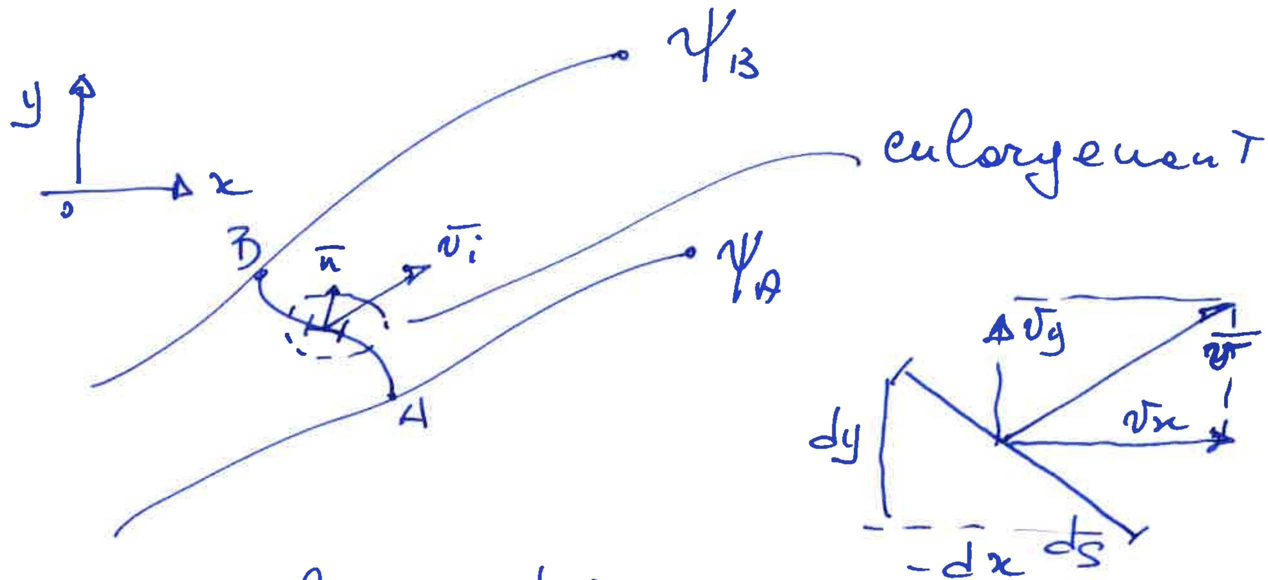
If we consider a streamline ($\psi = cost$)

$$\psi = cost \Rightarrow d\psi = 0 \quad \Rightarrow \quad \frac{v_x}{v_y} = \left.\frac{dx}{dy}\right|_{\psi = cost}$$



This is the local slope of the streamline

The difference in value between two streamlines is the flowrate actually flowing between the two streamlines.



w_{A-B} = length of segment A-B

$$\left.\frac{Q}{w}\right|_{A-B} = \int_{A-B} \bar{n}\cdot\bar{v}\,ds = \int_{A-B} (n_x v_x + n_y v_y)\,ds =$$

$\left[\frac{m^3}{m\,s}\right]$

$$= \int_{A-B} \overbrace{(n_x ds)}^{dx} v_x + \int_{A-B} \overbrace{(n_y ds)}^{dy} v_y =$$

$$= \int_{A-B} (v_x dy - v_y dx) = \int_{A-B} (-d\psi) =$$

$$= -(\psi_B - \psi_A) = \psi_A - \psi_B$$

Relations between vorticity ω, potential φ, and streamfunction ψ.

$$\omega_z \triangleq \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y \partial x} = 0$$

$$\omega_z \triangleq \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \Rightarrow \boxed{\overline{\omega} = \nabla^2 \psi}$$

Poisson Equation

These relations are valid in the case of potential, irrotational flow (which automatically satisfies the conditions of zero vorticity and continuity). The last unknown in this flow is pressure, which can be derived by the Navier-Stokes equations.

N/S (dimensionless) : $\frac{\partial \overline{v}}{\partial t} + (\overline{v}\cdot\overline{\nabla})\overline{v} = -\overline{\nabla} P + \frac{1}{Re}\nabla^2 \overline{v}$

Since we are in the limit $Re \to \infty$ the equation is written as:

$$\frac{\partial \overline{v}}{\partial t} + (\overline{v}\cdot\overline{\nabla})\overline{v} + \overline{\nabla} P = \cancel{\frac{1}{Re}\nabla^2 \overline{v}} \to 0$$

and if the flow is steady:

$$\underbrace{\rho(\bar{v}\cdot\bar{\nabla})\bar{v}}_{\bar{\nabla}\left(\frac{1}{2}\bar{v}\cdot\bar{v}\right) - \cancel{\bar{v}\times\bar{\omega}}} + \underbrace{\bar{\nabla}P}_{P = p + \rho g h} = 0$$

in Dimensional form

$\bar{v}\times\bar{\omega} = 0$: irrotational flow

$$\rho\bar{\nabla}\left(\tfrac{1}{2}\bar{v}\cdot\bar{v}\right) + \bar{\nabla}(p + \rho g h) = 0$$

$$\boxed{\cancel{\rho}\,\bar{\nabla}\left(\tfrac{1}{2}\rho v^2 + p + \rho g h\right) = \text{cost}}$$

Bernoulli Equation

This is indeed how Bernoulli derived this equation, not from the energy balance equation ~~Energy was not~~

The Bernoulli equation is valid along one streamline. Changing the line the constant changes.

Note on the vorticity equation and on Boundary conditions.

The 2D vorticity equation is 4° order equation

(A) $\bar{v} \cdot \bar{\nabla} \omega = \frac{1}{Re} \nabla^2 \omega$ (B) $[\omega = \nabla^2 \varphi]$

However, eq. (B) is 2° order and describes the flowfield.
Therefore, usual B.C. cannot be applied

$\bar{v} \cdot \bar{n} = 0$ no-cross condition

$\bar{v} \cdot \bar{t} = 0$ no-slip condition

$\bar{t}$ = Tangent versor to the surface
$\bar{n}$ = normal " " " "

The no-slip condition is thus redundant and not applied