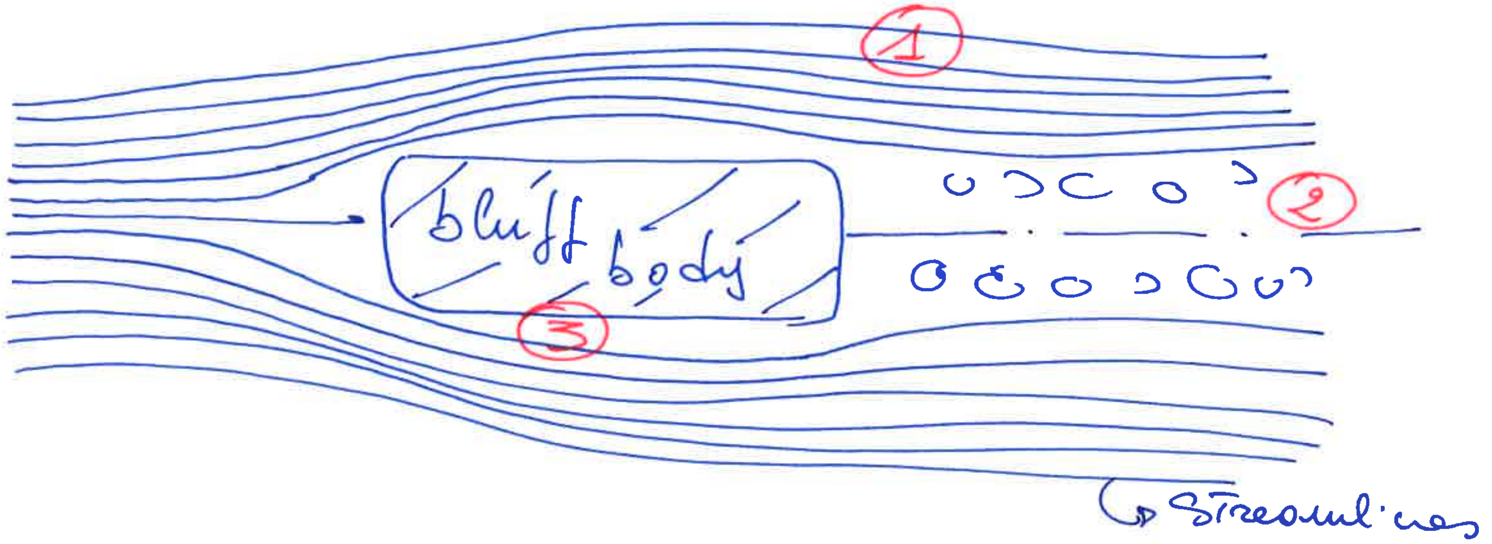
Vorlesung 6

Potential Flow

When we consider the flow of a fluid (with density ρ and viscosity μ) ~~~~ past a bluff body we can analyze the flow field in the following way:



We can identify the three regions

1. Potential Flow - Far from the body but with deformation of the streamlines
   - Negligible viscous dissipation
   - Vorticity $= 0$

2. Wake region (characterized by vortex shedding).

- Negligible viscous dissipation
- Vorticity non zero

3. Wall Region (Boundary Layer)

- Important viscous dissipation
- Vorticity Non zero

The velocity gradient near the wall is important due to the No-Slip Boundary condition. We observe high energy dissipation due to the big viscous stress

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

Definition of Vorticity

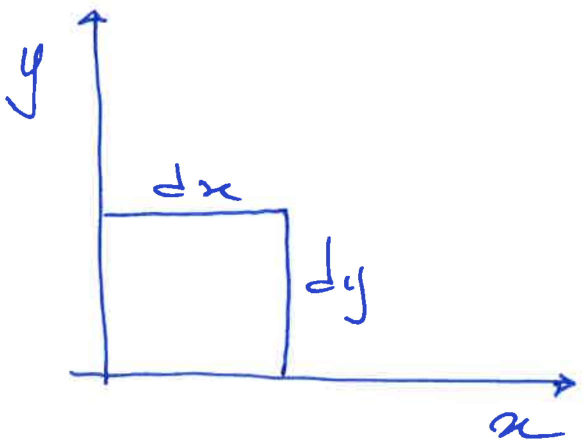
$$\vec{\omega} = \mathrm{rot}\,\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} =$$

$$= \vec{i}\underbrace{\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)}_{\omega_x} + \vec{j}\underbrace{\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)}_{\omega_y} + \vec{k}\underbrace{\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)}_{\omega_z}$$

Vorticity represent the local rotation rate of an elementary parcel of fluid. Since vorticity is defined by the derivatives of the velocity vector, it is also related to the deformation rate.

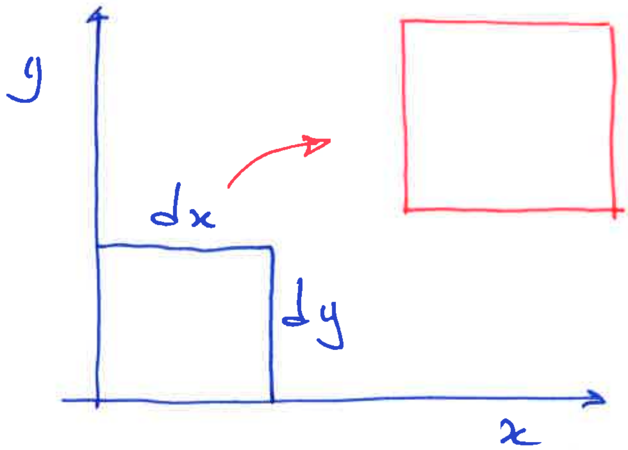
'Not to lecture'

- Definition of vorticity from the Rotation rate → kinematic derivation.

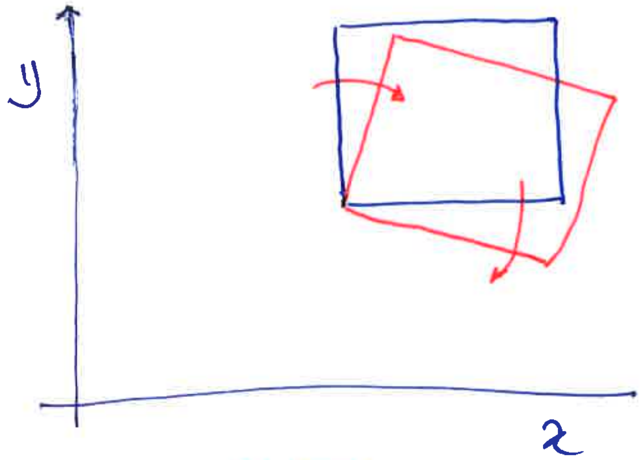


To simplify, we consider a 2D elemental fluid volume.

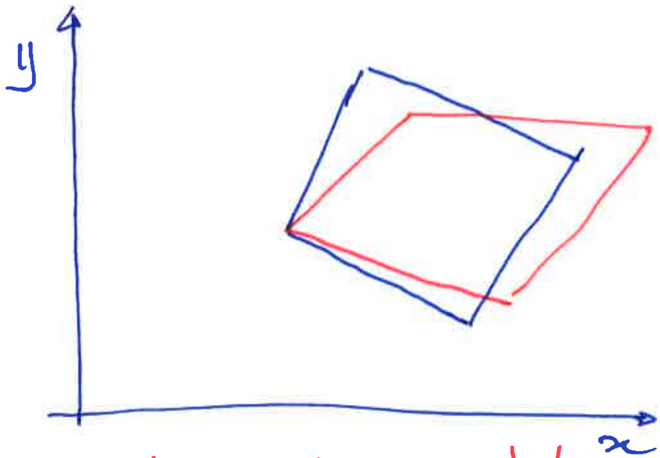
The elemental volume undergoes translation, rotation and angular deformation



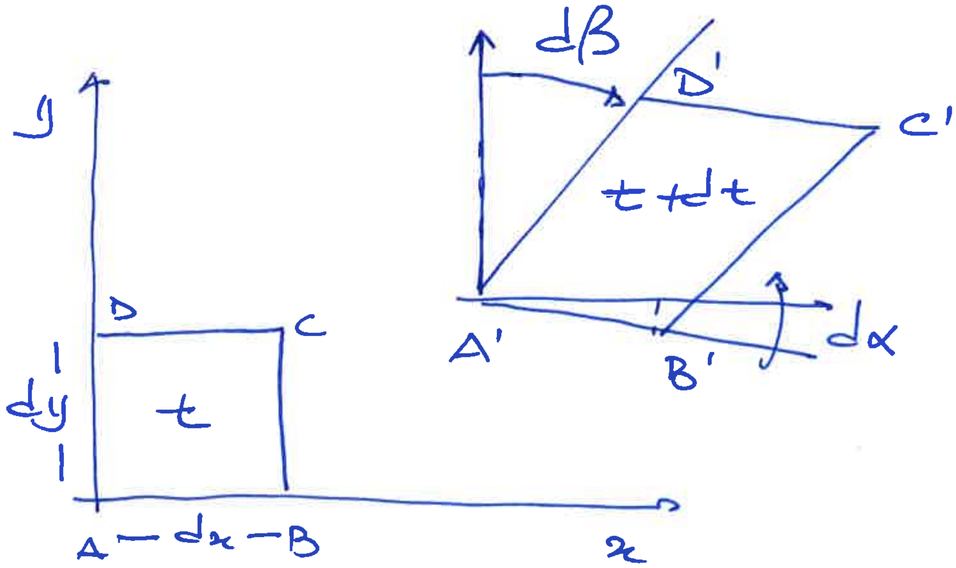
Translation



Rigid Rotation



Angular deformation



$d\beta \to$ clockwise ; $d\alpha \to$ counterclockwise

$$d\alpha = \tan^{-1}\left[\frac{(A'B')_y}{(A'B')_x}\right]$$

$$(A'B')_y = \left[v_y\Big|_{x+dx} dt - v_y\Big|_{x} dt \right]$$

$$(A'B')_x \cong dx \qquad \Leftarrow d\alpha <<$$

in the limit of $d\alpha$ very small

Substituting : $d\alpha = \tan^{-1}\left[\frac{\partial v_y}{\partial x}\frac{dx}{dx}\, dt\right] \simeq$

$$\simeq \frac{\partial v_y}{\partial x}\, dt$$

$$\Rightarrow \quad \frac{d\alpha}{dt} = \dot{\alpha} = \frac{\partial v_y}{\partial x}$$

Analogous is the derivation of $\dot{\beta}$

$$\frac{d\beta}{dt} = \dot{\beta} = \frac{\partial v_x}{\partial y}$$

Therefore, the ~~xxxx~~ mean rotation rate (counterclockwise) is:

$$\frac{1}{2}\left[\dot{\alpha} - \dot{\beta}\right] = \frac{1}{2}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]$$

Vorticity is related to the mean rotation rate. In 2D

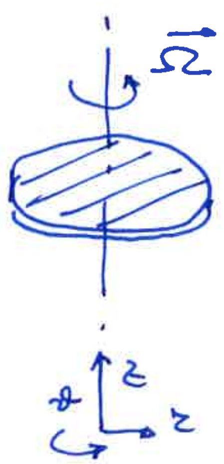
$$\omega = 2 \cdot \frac{1}{2}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right] = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

$\boxed{\text{Not to lecture}}$

We can now try to make some examples to understand better the role of vorticity:

$\boxed{1}$ Example: a fluid is rotating as if it were a rigid body with angular rotation rate Ω.

The velocity field is $\vec{u} = \vec{\Omega} \times \vec{r}$



$$\vec{\Omega} = \Omega_z \hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{v} = \Omega_z \hat{k} \times (x\hat{i} + y\hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega_z \\ x & y & 0 \end{vmatrix} =$$

$$= \underbrace{\Omega_z \cdot x}_{v_y}\,\hat{j} - \underbrace{\Omega_z\, y}_{v_x}\,\hat{i} \qquad \text{and } v_z = 0$$

The components of vorticity are:

$$\omega_x = \cancel{\frac{\partial v_z}{\partial y}}^{\,0} - \frac{\partial v_y}{\partial z} = -\frac{\partial}{\partial z}\big(\underbrace{\Omega_z \cdot x}_{\text{const.}}\big) = 0$$

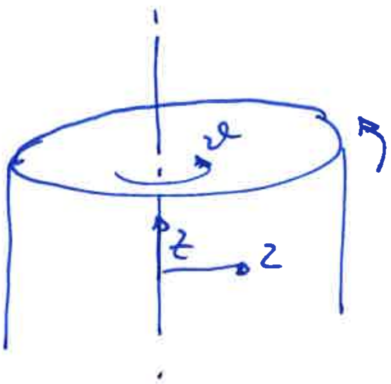
$$\omega_y = \frac{\partial v_x}{\partial z} - \cancel{\frac{\partial v_z}{\partial x}}^{\,0} = \frac{\partial}{\partial z}\big(\underbrace{\Omega_z \cdot y}_{\text{const}}\big) = 0$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial}{\partial x}\left(-\Omega_z y\right) - \frac{\partial}{\partial y}\left(\Omega_z x\right) =$$

$$= 2\,\Omega_z \neq 0$$

$$\vec{\omega} = \omega_z\,\hat{k}$$ [orthogonal to the motion plane]

12) Example: Every fluid particle is moving on a circular path about the z-axis but with the radial velocity distribution corresponding to the torsional flow.



$$v_\theta = \frac{k}{r} \quad \text{with } k = \text{constant}$$

it is the flow generated ~~by a~~ in a cylindrical container by the boundary which moves at constant speed.

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_\theta) - \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}^{\,0} =$$

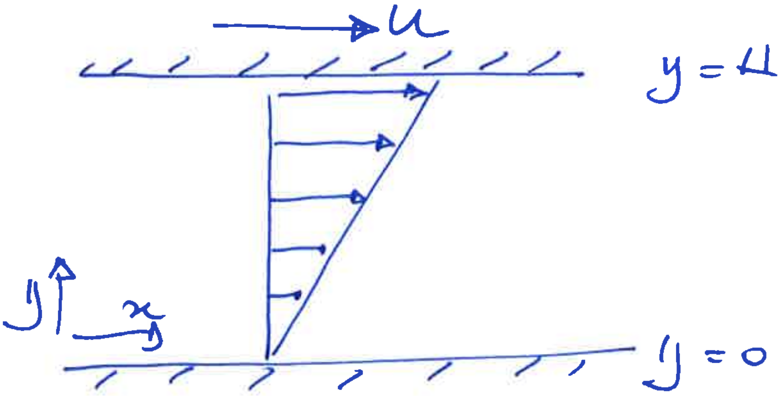
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{k}{r}\right) = \phi$$

with ω_r and $\omega_\theta = 0$

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = \phi$$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0$$

$\boxed{13}$ <u>EXAMPLE</u> This is the simple shear flow in a Couette device.



$$v_x(y) = \frac{U}{H}\, y$$

$$\omega_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0$$

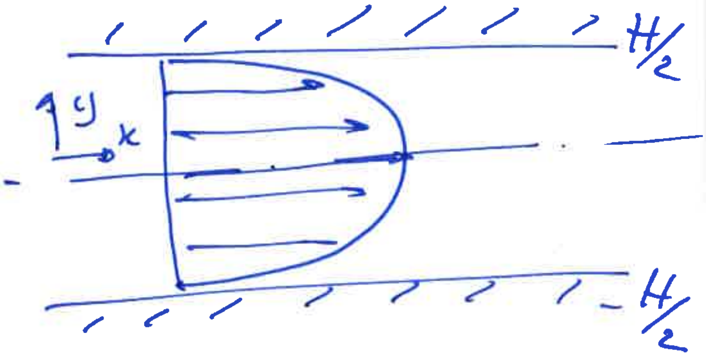
$$\omega_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{U}{H}$$

$\frac{U}{H}$ is, of course, the slope of the velocity profile.

14 Example - Plane Poiseuille flow

$$v_x(y) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)\left[y^2 - \frac{H^2}{4}\right]$$

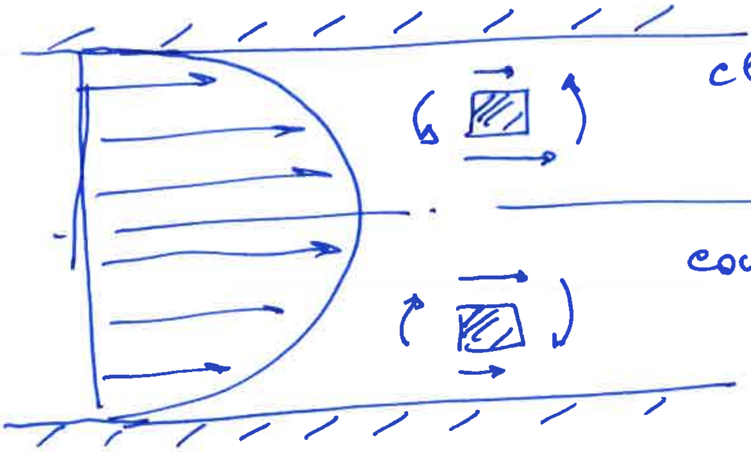


$$\omega_x = \omega_y = 0$$

$$\omega_z = -\frac{\partial v_x}{\partial y} =$$

$$= -\frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)2y = -\frac{1}{\mu}\frac{\Delta P}{L}y$$

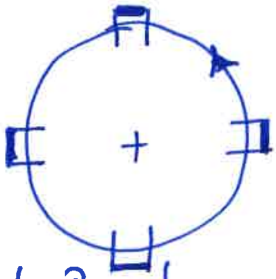
maximum vorticity (magnitude) is at both walls. Vorticity is zero in the centerplane.



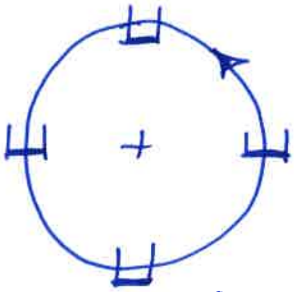
clockwise, negative vorticity

counterclockwise, positive vorticity

Discussion Examples $\boxed{11}$ & $\boxed{12}$



Rigid Body Rotation



Circulation w/o rotation

in the limit of $\sqcup$ very small