# Application of $k - \mathcal{E}$ model to channel flow

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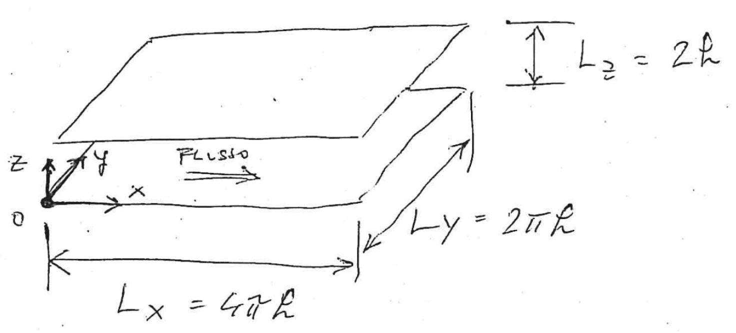
# Physical problem and governing equations

We consider the turbulent channel flow at $Re_* = 150$ ($\equiv h^+$):




Flow conditions: steady ($\frac{\partial \cdot}{\partial t} = 0$), 2D ($\frac{\partial \cdot}{\partial y} = 0$) and unidirectional ($\overline{u} \neq 0$, $\overline{v} = \overline{w} = 0$)

# Physical problem and governing equations

Governing equations:

- Continuity: $$\frac{\partial \overline{u}}{\partial x} = 0$$
- RANS (x-comp): $$0 = -\frac{\partial \overline{P}}{\partial x} + \mu \frac{\partial^2 \overline{u}}{\partial z^2} + \frac{\partial}{\partial z}\left(\rho C_\mu \frac{k^2}{\mathcal{E}} \frac{\partial \overline{u}}{\partial z}\right)$$
- Transport equation for the TKE:
$$0 = P_k - \mathcal{E} + \frac{\partial}{\partial z}\left(\frac{C_\mu}{\sigma_k} \frac{k^2}{\mathcal{E}} \frac{\partial k}{\partial z}\right)$$
- Transport equation for $\mathcal{E}$:
$$0 = C_{\mathcal{E}1} P_k \frac{\mathcal{E}}{k} - C_{\mathcal{E}2} \frac{\mathcal{E}^2}{k} + \frac{\partial}{\partial z}\left(\frac{\nu^t}{\sigma_{\mathcal{E}}} \frac{\partial \mathcal{E}}{\partial z}\right)$$

with $\nu^t = C_\mu \frac{k^2}{\mathcal{E}}$ and $P_k = -\rho \overline{u' w'} \frac{\partial \overline{u}}{\partial z} = -\nu^t \left(\frac{\partial \overline{u}}{\partial z}\right)^2$.

# Physical problem and governing equations

Dimensionless equations (in wall units):

- Continuity: $\dfrac{\partial \overline{u}^+}{\partial x^+} = 0$
- RANS (x-comp): $0 = -\dfrac{\partial \overline{P^+}}{\partial x^+} + \dfrac{\partial^2 \overline{u}^+}{\partial {z^+}^2} + \dfrac{\partial}{\partial z^+}\left( C_\mu \dfrac{{k^+}^2}{\mathcal{E}^+} \dfrac{\partial \overline{u}^+}{\partial z^+} \right)$
- Transport equation for the TKE:

$$0 = C_\mu \frac{{k^+}^2}{\mathcal{E}^+} \left( \frac{\partial \overline{u}^+}{\partial z^+} \right)^2 - \mathcal{E}^+ + \frac{\partial}{\partial z^+} \left( \frac{C_\mu}{\sigma_k} \frac{{k^+}^2}{\mathcal{E}^+} \frac{\partial k^+}{\partial z^+} \right)$$

- Transport equation for $\mathcal{E}$:

$$0 = C_{\mathcal{E}1} C_\mu k^+ \left( \frac{\partial \overline{u}^+}{\partial z^+} \right)^2 - C_{\mathcal{E}2} \frac{{\mathcal{E}^+}^2}{k^+} + \frac{\partial}{\partial z^+} \left( \frac{C_\mu}{\sigma_{\mathcal{E}}} \frac{{k^+}^2}{\mathcal{E}^+} \frac{\partial \mathcal{E}^+}{\partial z^+} \right)$$

# Physical problem and governing equations

where:

- $\overline{u}^+ = \dfrac{\overline{u}}{u_*}$
- $z^+ = \dfrac{z \cdot u_*}{\nu}$ (note: $z = h \rightarrow z^+ = Re_*$)
- $k^+ = \dfrac{k}{u_*^2}$
- $\mathcal{E}^+ = \dfrac{\mathcal{E} \cdot \nu}{u_*^4}$
- $\overline{P}^+ = \dfrac{\overline{P}}{\rho u_*^2}$

Note: $\dfrac{\partial \overline{P}^+}{\partial x^+} = \dfrac{1}{Re_*} \dfrac{\partial \overline{P}^-}{\partial x^-}$ with $\overline{P}^- = \overline{P}^+$, $x^- = \dfrac{x}{h}$ and $\dfrac{\partial \overline{P}^-}{\partial x^-} = -1$.

# Physical problem and governing equations

The systems of dimensionless PDEs on slide 4 can be written as a system of ODEs by putting:

- $X_1 := k^+ \quad \rightarrow \quad X_3 := \dfrac{\partial k^+}{\partial z^+}$
- $X_2 := \mathcal{E}^+ \quad \rightarrow \quad X_4 := \dfrac{\partial \mathcal{E}^+}{\partial z^+}$
- $X_5 := \overline{u}^+ \quad \rightarrow \quad X_6 := \dfrac{\partial \overline{u}^+}{\partial z^+}$

Performing this change of variables yields the following system of 6 ODEs:

# Physical problem and governing equations

- $\dfrac{\partial X_1}{\partial z^+} = X_3$
- $\dfrac{\partial X_2}{\partial z^+} = X_4$
- $\dfrac{\partial X_5}{\partial z^+} = X_6$
- RANS:

$$\frac{\partial X_6}{\partial z^+} = \left[\frac{\partial \overline{P}^+}{\partial x^+} - C_\mu X_6 \left(\frac{2X_1X_3}{X_2} - \frac{X_1^2 X_4}{X_2^2}\right)\right] / \left(1 + C_\mu \frac{X_1^2}{X_2}\right)$$

- Transp. TKE:

$$\frac{\partial X_3}{\partial z^+} = -\frac{2X_3^2}{X_1} + \frac{X_3X_4}{X_2} - \sigma_k X_6^2 + \frac{\sigma_k}{C_\mu}\left(\frac{X_2}{X_1}\right)^2$$

- Transp. $\mathcal{E}$:

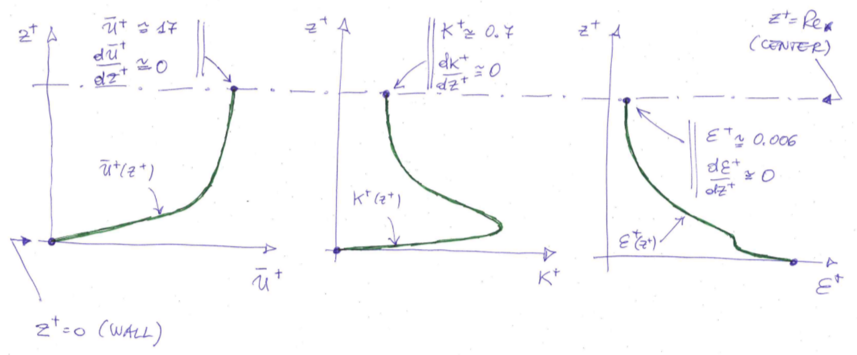
$$\frac{\partial X_4}{\partial z^+} = -\frac{2X_4X_3}{X_1} + \frac{X_4^2}{X_2} - C_{\mathcal{E}1}\sigma_{\mathcal{E}} X_6^2 \cdot \frac{X_2}{X_1} + \frac{C_{\mathcal{E}2}\sigma_{\mathcal{E}}}{C_\mu}\left(\frac{X_2}{X_1}\right)^2$$

Note: RANS rewritten wrt $\partial\overline{u}^+/\partial z^+$, Tr. TKE wrt $\partial\overline{k}^+/\partial z^+$, etc.

# Boundary conditions

The ODEs can be solved upon integration along $z^+$. To perform such integration B.C.s for $X_1$, ... , $X_6$ are needed.

From DNS, we know that:




Hence, at $z^+ = Re_*$:

$X_1 = 0.7$, $X_2 = 0.006$, $X_3 = X_4 = 0$, $X_5 = 17$, $X_6 = 0$ .

# Solution with 6 ODEs

To solve the full system of ODEs, one must solve an **Initial Value Problem** (IVP), not to be confused with a **Boundary Value Problem** (BVP)[*].

To this aim, one of the solvers of Matlab® can be used, e.g. ODE45 (which is based on an explicit $4^{th}/5^{th}$ order Runge-Kutta scheme) for the equations involving $k$ and $\mathcal{E}$, or ODE113 (which is based on a Adams-Bashforth-Moulton scheme) better suited for the equations involving $\overline{u}$.

[*] Given $f(t)$ with $0 < t < 1$: IVP specifies $f(t=0)$ and $f'(t=0)$, BVP specifies $f(t=0)$ and $f(t=1)$.

## Solution with 4 ODEs

Instead of solving the full system of ODEs, one can solve only the first 4 ODEs and obtain $\frac{\partial \overline{u}^+}{\partial z^+}$ directly from:

$$\overline{u}^+ \equiv X_5 = \begin{cases} z^+ & \text{for } \; z^+ \leq 11.6 \\ 2.5 \ln z^+ + 5.5 & \text{for } \; 11.6 \leq z^+ \leq Re_* \\ \text{symmetric} & \text{for } \; z^+ \geq Re_* \end{cases}$$

which gives:

$$\frac{\partial \overline{u}^+}{\partial z^+} \equiv X_6 = \begin{cases} 1 & \text{for } \; z^+ \leq 11.6 \\ 2.5/z^+ & \text{for } \; 11.6 \leq z^+ \leq Re_* \\ \text{symmetric} & \text{for } \; z^+ \geq Re_* \end{cases}$$

without the need to integrate the equations for $X_5$ and $X_6$.

# Notes on the solution

1) When solving the full system of ODEs, one must impose the value of $\frac{\partial \overline{P}^+}{\partial x^+} = \frac{1}{Re_*}\frac{\partial \overline{P}^-}{\partial x^-}$. Since $\frac{\partial \overline{P}^-}{\partial x^-} = -1 \rightarrow \boxed{\frac{\partial \overline{P}^+}{\partial x^+} \simeq -0.00\overline{6}}$.

2) Possible "tricks" to improve the solution:

- Use:

$$\frac{\partial P^+}{\partial z^+} = \begin{cases} -8 \cdot 10^{-7} & \text{for } z^+ \leq 30 \\ -2 \cdot 10^{-8} & \text{for } 30 \leq z^+ \leq Re_* \\ \text{symmetric} & \text{for } z^+ \geq Re_* \end{cases}$$

- Use: $\frac{\partial k}{\partial z^+} \simeq -1.84 \cdot 10^{-2}$ instead of 0 and/or

$$\frac{\partial \overline{u}^+}{\partial z^+} = \begin{cases} 0.11 & \text{for } z^+ \leq 15 \\ 2.46/z^+ & \text{for } 15 \leq z^+ \leq Re_* \\ \text{symmetric} & \text{for } z^+ \geq Re_* \end{cases}$$

- Change value of model's constants.