Flow over a sphere: force calculation

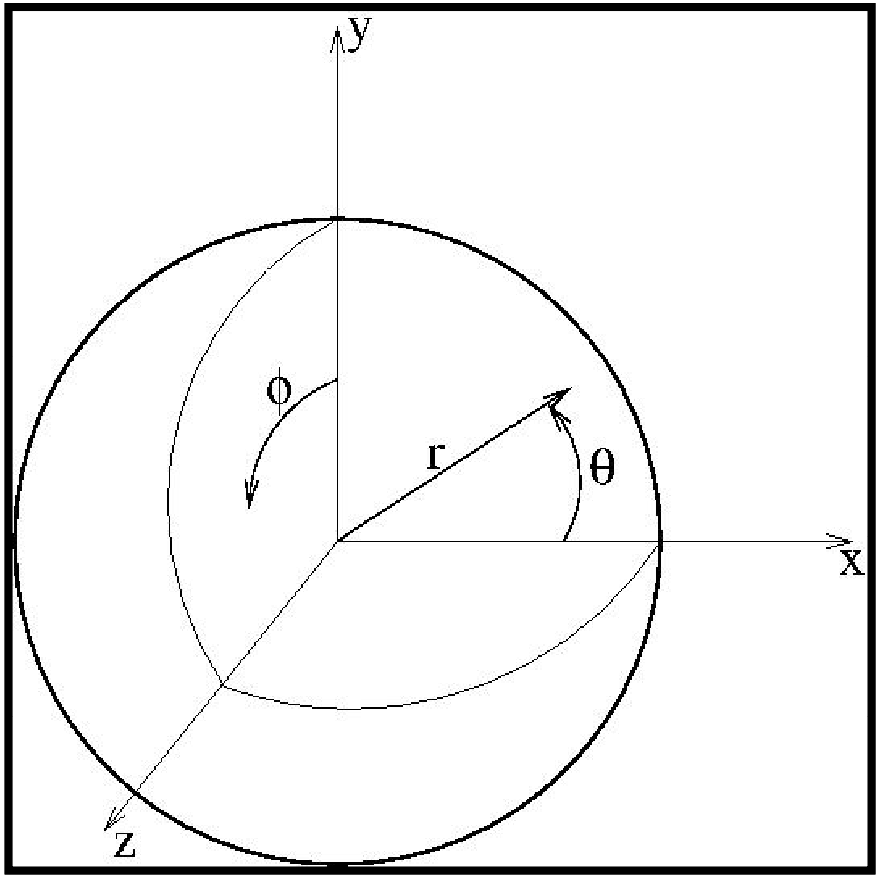
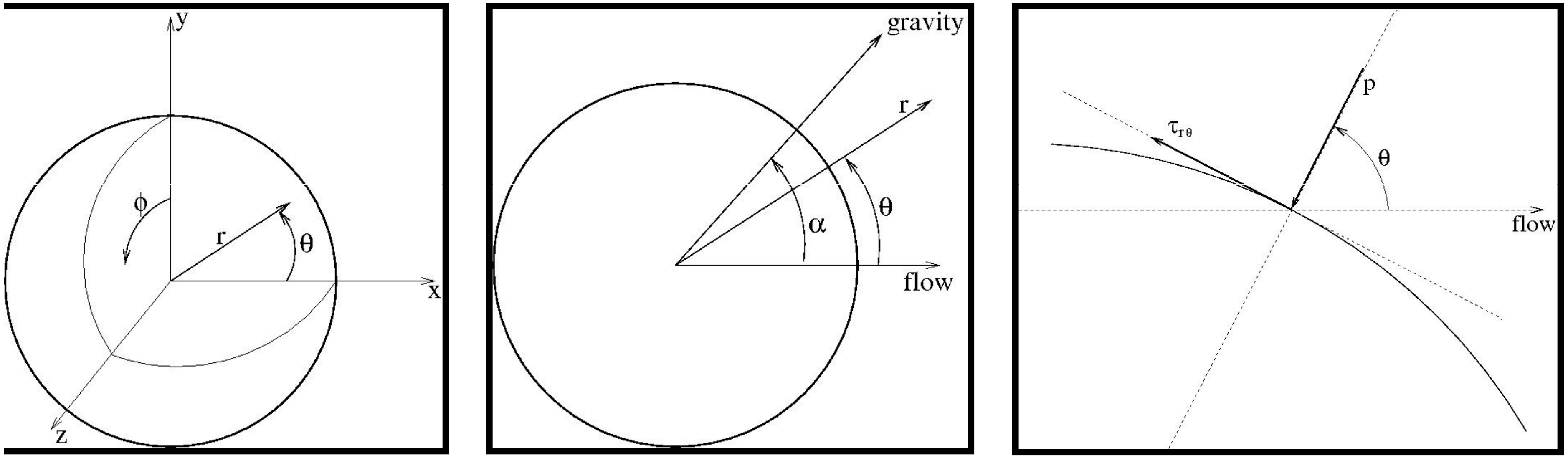


Fig.1: Reference frame

Velocity and pressure field

$$
\begin{cases}
u_r = U\left[1-\dfrac{3}{2}\left(\dfrac{R}{r}\right)+\dfrac{1}{2}\left(\dfrac{R}{r}\right)^3\right]\cos\theta \\
u_\theta = -U\left[1-\dfrac{3}{4}\left(\dfrac{R}{r}\right)-\dfrac{1}{4}\left(\dfrac{R}{r}\right)^3\right]sen\,\theta \\
P-P_0 = \left[-\dfrac{3}{2}\mu\left(\dfrac{U}{R}\right)\left(\dfrac{R}{r}\right)^3\right]\cos\theta
\end{cases}
$$

Flow over a sphere: force calculation



Helpful sketches (fig 2,3,4 above)

Expression of the force on the sphere surface

$$F=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-p\cos\theta-\tau_{r\theta}sen\theta\right)R^2 sen\theta\cdot d\theta d\phi$$

Expression of surface stresses

$$\begin{cases}\tau_{r\theta}=-\dfrac{3}{2}\dfrac{\mu U sen\theta}{R}\\ p=P_0-\rho g r\left(\cos\alpha\cdot\cos\theta+sen\alpha\cdot sen\theta\right)-\dfrac{3}{2}\mu\dfrac{U}{R}\left(\dfrac{R}{r}\right)^3\cos\theta\end{cases}$$

With some algebra….

Flow over a sphere: force calculation

$$F = -2\pi P_0 R^2 \int_0^{\pi} sen\theta \cos\theta d\theta +$$

$$+ 2\pi\rho g R^3 \cos\alpha \int_0^{\pi} sen\theta \cos^2\theta d\theta +$$

$$+ 2\pi\rho g R^3 sen\alpha \int_0^{\pi} sen^2\theta \cos\theta d\theta +$$

$$+ 3\pi\mu U R \int_0^{\pi} sen\theta \cos^2\theta d\theta +$$

$$+ 3\pi\mu U R \int_0^{\pi} sen^3\theta d\theta$$

Calculating the value of the integrals we obtain:

$$F = \frac{4}{3}\pi\rho g R^3 \cos\alpha + 2\pi\mu U R + 4\pi\mu U R$$

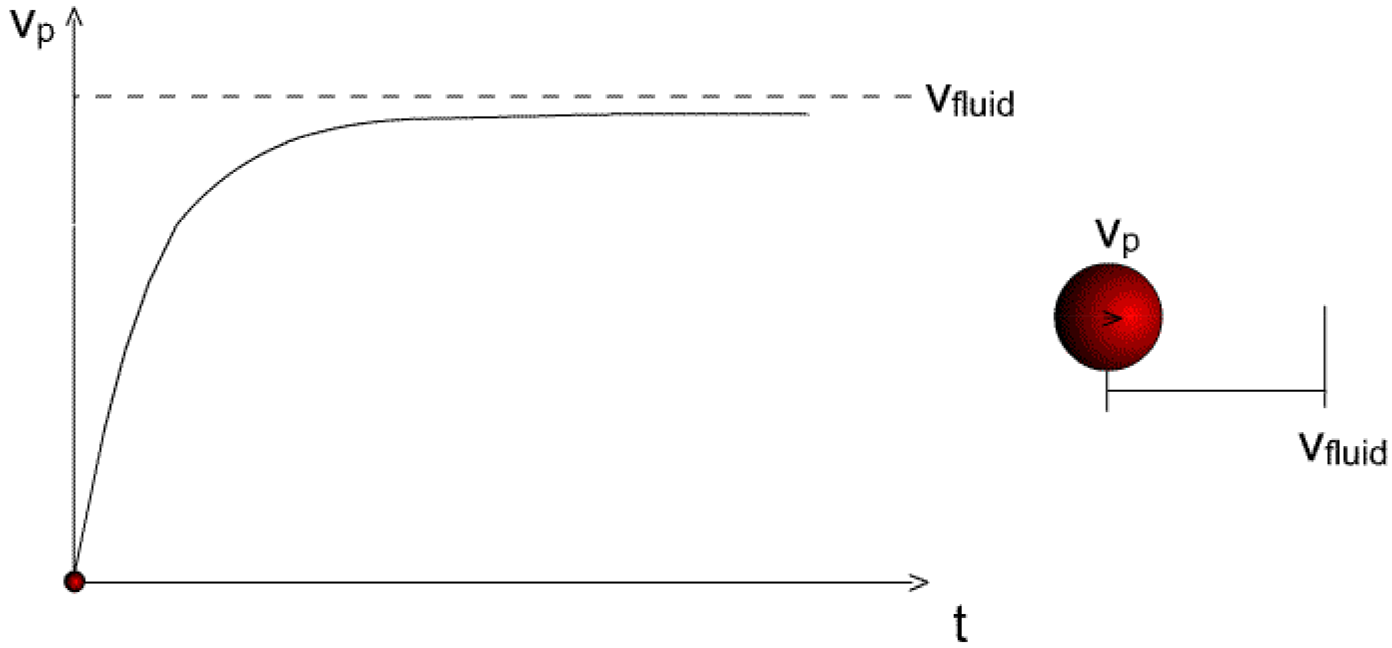
buoyancy (first term), *Form drag* (second term), *Skin friction* (third term)

Total drag force (no buoyancy):

$$F = 6\pi\mu U R$$

Stokes law

Is it the only thing?



Added Mass Force

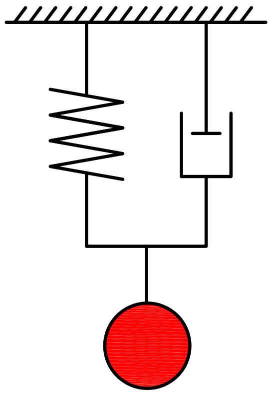
- A Body moving into a fliud produces motion of the surrounding the body

- Accelerating a body requires the acceleration of a fraction of the fluid around the body
- Added Mass:

$$m_p \vec{a} = \vec{F} \quad \Rightarrow \quad (m_p + m_a)\vec{a} = \vec{F}$$

- F. Bessel, 1828
- Odar e Hamilton, 1964

} Harmonic motion of a sphere



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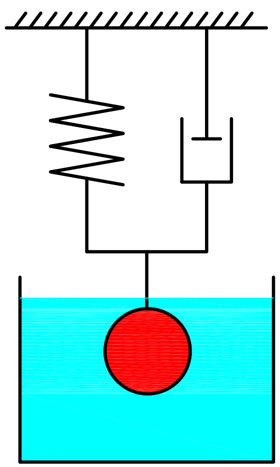

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$$T' = 2\pi\sqrt{\frac{m}{k}} \quad , \quad T'' = 2\pi\sqrt{\frac{m + m_a}{k}}$$

Added Mass force: derivation

- Spherical body in motion with velocity $U_{(t)}$
- Potential Flow Theory (Incompressible, inviscid and irrotational flow)

$$\left.\begin{aligned}\nabla\cdot\vec{u}=0\\ \vec{u}=-\nabla\varphi\end{aligned}\right\}\Rightarrow\nabla^2\varphi=0\ ,\ \varphi=-\frac{UR^3}{2r^2}\cos\vartheta$$

- Kinetic Energy of the fluid

$$T=\int_V\left(\frac{1}{2}\rho_F u^2\right)dV=\frac{1}{2}\rho_F\int_V(\nabla\varphi\cdot\nabla\varphi)dV=\ldots=\frac{\pi\rho_F R^3U_{(t)}^2}{3}$$

$$UF_{added\,mass}=\frac{dT}{dt}=\frac{1}{2}\left(\frac{4\pi\rho_F R^3}{3}\right)U\frac{dU}{dt}=U\frac{1}{2}m_F\frac{dU}{dt}$$

Added Mass Force

- Spherical particle
- Potential Flow

$$\left.\right\} \Rightarrow \quad F_{added\,mass} = \frac{1}{2} m_F \frac{dU}{dt}$$

$$F_{added\,mass} \neq 0 \Leftrightarrow \frac{dU}{dt} \neq 0$$

- Virtual density (apparent density)

$$\boxed{\rho_{virt} = \rho_p + \frac{1}{2}\rho_F}$$

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- The inertia of the particle increases