We can therefore write

$$\frac{\text{inertial terms}}{\text{viscous terms}} \sim \frac{\rho r \Omega^2}{\eta r \Omega / H^2} = \frac{H^2 \Omega \rho}{\eta} \ll 1 \tag{12.19}$$

This condition is easily satisfied in practice for viscometers. We require separately that $R/H \gg 1$, for we have neglected any deviations from the circular flow pattern in the neighborhood of the liquid–air interface at the edge.

12.3 FLOW AROUND A SPHERE

12.3.1 Problem Formulation

We studied flow around a submerged sphere in Chapter 4, where we found that for Re < 1 the drag coefficient varies inversely with Reynolds number [Eq. (4.3)]. This was interpreted as a regime of negligible inertial forces. We will show here that this result, known as *Stokes' law*, can be derived from the creeping flow equations.

The flow is shown schematically in Fig. 12-3. A sphere of diameter D is submerged in an infinite body of fluid. The fluid moves past the sphere uniformly with a velocity U; that is, an observer situated at any point on the surface of the sphere will see a uniform flow with velocity U at a large distance from the sphere, although the velocity near the sphere will differ from the constant value. (We have changed nomenclature here from that used in Chapter 4. The subscript p to denote particle has not been retained, and the relative velocity between the sphere and the surrounding fluid has been changed from V_p to U. U is commonly used in the fluid mechanics literature to denote a velocity far from, and unaffected by, a solid surface.)

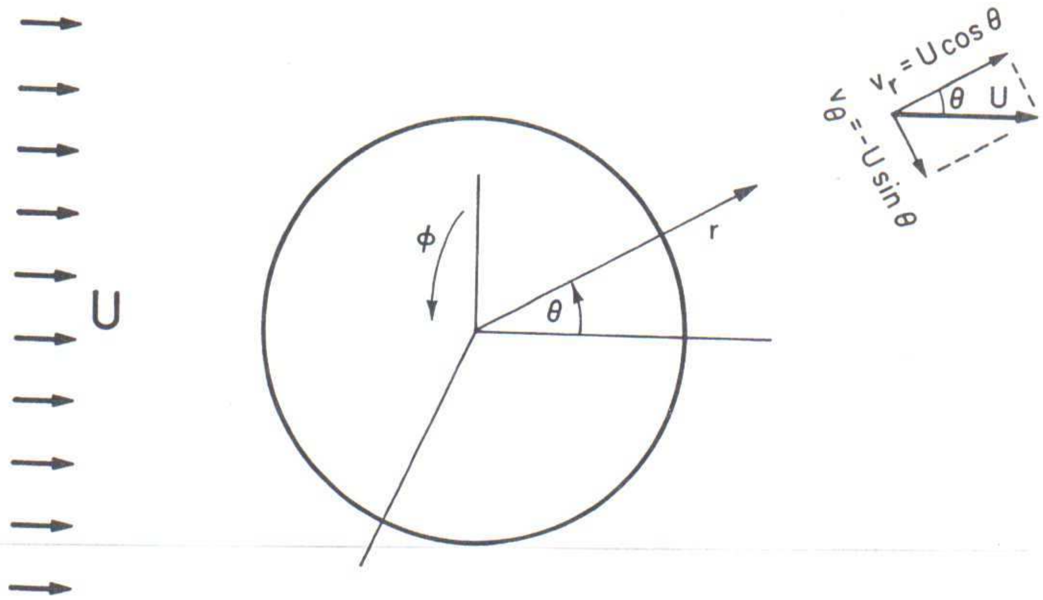

Figure 12-3. Schematic of uniform flow past a sphere.

The characteristic velocity for this flow is clearly the relative uniform velocity, U, and the characteristic length is the sphere diameter, D. These choices are unambiguous, because U and D are the only velocity and length specified in the problem. The Reynolds number is therefore as defined in Eq. (4.1a),

$$\text{Re} = \frac{DU\rho}{\eta} \tag{12.20}$$

The condition for applicability of the creeping flow approximation is, of course, $\text{Re} \ll 1$.

The flow is best described in a spherical coordinate system, as shown in Fig. 12-3. We will assume ϕ-direction symmetry, $\partial/\partial\phi = 0$, and $v_\phi = 0$. Then the continuity equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) = 0 \tag{12.21}$$

The r and θ components of the creeping flow approximation to the Navier–Stokes equations are obtained from Table 7-10 by setting $\rho = 0$, as follows:

$$0 = -\frac{\partial\mathcal{P}}{\partial r} + \eta\Big[\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial v_r}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial v_r}{\partial\theta}\Big) - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2}v_\theta\cot\theta\Big] \tag{12.22a}$$

$$0 = -\frac{1}{r}\frac{\partial\mathcal{P}}{\partial\theta} + \eta\Big[\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial v_\theta}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\cdot\Big(\sin\theta\frac{\partial v_\theta}{\partial\theta}\Big) + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\Big] \tag{12.22b}$$

The no-slip boundary condition at the sphere surface $r = R$ requires

$$r = R: \quad v_r = v_\theta = 0 \tag{12.23a}$$

The boundary condition far from the sphere is obtained from the requirement that the flow approach the uniform velocity, U. When the velocity vector at large r is resolved into r and θ components, as shown in Fig. 12-3, we obtain the condition

$$r \longrightarrow \infty: \quad v_r = U\cos\theta \qquad v_\theta = -U\sin\theta \tag{12.23b}$$

12.3.2 Solution of Equations

The solution of Eqs. (12.21), (12.22), and (12.23) is suggested by the form of the boundary condition at infinity. The boundary conditions usually provide insight into the structure of the solution. In this case we see that the θ dependence far from the sphere is given by a cosine and sine relation. It seems reasonable to look for a solution that retains this simple angle dependence over the entire field. Thus, we are motivated to seek a solution in the form

$$v_r = A(r)\cos\theta \qquad (12.24a)$$

$$v_\theta = B(r)\sin\theta \qquad (12.24b)$$

$A(r)$ and $B(r)$ are unknown functions of r, but comparison with Eqs. (12.23) shows that A and B must satisfy the following conditions at $r = R$ and $r \to \infty$:

$$r = R: \quad A(R) = B(R) = 0 \qquad (12.25a)$$

$$r \longrightarrow \infty: \quad A(\infty) = U \qquad B(\infty) = -U \qquad (12.25b)$$

The remainder of this section is given over to the use of Eqs. (12.21) and (12.22) to find the functions $A(r)$ and $B(r)$. Readers who are not interested in the details may wish to skip to the solution, Eqs. (12.34) and (12.35).

Substitution of Eqs. (12.24) into the continuity equation, Eq. (12.21), yields

$$\frac{\cos\theta}{r^2}\left[2rA(r) + r^2\frac{dA(r)}{dr}\right] + \frac{2\cos\theta}{r}B(r) = 0 \qquad (12.26a)$$

Dividing by $2\cos\theta/r$ eliminates the θ dependence and gives an equation involving only A and B,

$$A(r) + \frac{r}{2}\frac{dA}{dr} + B(r) = 0 \qquad (12.26b)$$

This equation shows that the assumed form of solution is consistent with continuity, and furthermore continuity reduces the number of unknown variables by showing how A and B must be related.

We now turn to the r component of the creeping flow equations, Eq. (12.22a). When the form of the solution given by Eqs. (12.24) is substituted into this component equation, we obtain an equation that can be written

$$\frac{\partial \mathcal{P}}{\partial r} = \eta\,(\text{function of } r)\cos\theta \qquad (12.27)$$

This relation requires that $\mathcal{P}$ be of the form

$$\mathcal{P} = \mathcal{P}_0 + \eta\Pi(r)\cos\theta \qquad (12.28)$$

$\Pi(r)$ is an unknown function of r. $\mathcal{P}_0$ must be independent of r, although it could depend on θ. Substitution of Eq. (12.28) into Eq. (12.22a) then gives a relation among Π, A, and B:

$$0 = -\frac{d\Pi(r)}{dr} + \frac{1}{r^2}\frac{d}{dr}r^2\frac{dA(r)}{dr} - \frac{4A(r)}{r^2} - \frac{4B(r)}{r^2} \qquad (12.29)$$

We must now satisfy the θ component of the creeping flow equations, Eq. (12.22b). Note that $\mathcal{P}$, v_r, and v_θ are now expressed in terms of the functions $\Pi(r)$, $A(r)$, and $B(r)$, respectively, with specified θ dependence, so there are no remaining degrees of freedom. Thus, Eq. (12.22b) provides an important consistency check on the assumptions that we have made about the solution. When Eqs. (12.24) and (12.28) are substituted into Eq. (12.22b) we obtain, after some grouping of terms and multiplication by $r/\sin\theta$,

$$0 = \Pi(r) + \frac{1}{r}\frac{d}{dr} r^2 \frac{dB(r)}{dr} - \frac{2B(r)}{r} - \frac{2A(r)}{r} \tag{12.30}$$

Thus, the assumed θ dependence for v_r and v_θ does reduce the partial differential equations to a set of linear *ordinary* differential equations for the three functions of r, $\Pi(r)$, $A(r)$, and $B(r)$. It should be noted that it is also established in the derivation of Eq. (12.30) that $\mathcal{P}_0$ must be a true constant and not a function of θ.

The three linear ordinary differential equations, Eqs. (12.24), (12.29), and (12.30), are most easily solved by eliminating Π and B to obtain a single equation for A. Equation (12.30) is differentiated with respect to r to obtain an equation for $d\Pi/dr$, which is then substituted into Eq. (12.29). Equation (12.26b) is then used to eliminate B. The resulting fourth-order equation for $A(r)$, after some simplification, is

$$0 = r\frac{dA}{dr} - r^2\frac{d^2A}{dr^2} - r^3\frac{d^3A}{dr^3} - \frac{r^4}{8}\frac{d^4A}{dr^4} \tag{12.31}$$

Two of the boundary conditions for this fourth-order equation are given directly in Eqs. (12.25), and the other two are obtained by evaluating Eq. (12.26b) at $r = R$ and $r \longrightarrow \infty$ and using the conditions for B in Eqs. (12.25). The resulting four boundary conditions are thus

$$r = R: \quad A(R) = \frac{dA}{dr} = 0 \tag{12.32a}$$

$$r \longrightarrow \infty: \quad A(\infty) = U \qquad \frac{dA}{dr} = 0 \tag{12.32b}$$

Equation (12.31) is a special form of linear homogeneous equation that is known as *Euler's equation.** This equation has solutions of the form r^n. Substituting $A = r^n$ into Eq. (12.31) gives

$$r^n(-n^4 - 2n^3 + 5n^2 + 6n) = 0$$

n must be a root of the polynomial in parentheses. The roots are $n = 0, -1, -3, +2$, so the general solution to Eq. (12.31) is

$$A(r) = C_1 + C_2r^{-1} + C_3r^{-3} + C_4r^2$$

The conditions (12.32b) at $r \longrightarrow \infty$ require that $C_4 = 0$, $C_1 = +U$, while the conditions (12.32a) at $r = R$ become

$$A(R) = U + \frac{C_2}{R} + \frac{C_3}{R^3} = 0$$

$$\frac{dA(R)}{dr} = -\frac{C_2}{R^2} - \frac{3C_3}{R^4} = 0$$

*There are many equations named after Euler. Equation (12.31) should not be confused with the flow equations for an inviscid fluid, which are also known as Euler's equation.

These latter two equations have a solution $C_2 = -3UR/2$, $C_3 = +UR^3/2$. The complete solution to Eqs. (12.31) and (12.32) is then

$$A(r) = U\left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right] \tag{12.33a}$$

$B(r)$ and $\Pi(r)$ then follow from Eqs. (12.26b) and (12.30) as

$$B(r) = -U\left[1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3\right] \tag{12.33b}$$

$$\Pi(r) = \frac{-3U}{2R}\left(\frac{R}{r}\right)^2 \tag{12.33c}$$

Finally, the r and θ components of the velocity and the equivalent pressure are obtained from Eqs. (12.24) and (12.28):

$$v_r = U\left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta \tag{12.34a}$$

$$v_\theta = -U\left[1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta \tag{12.34b}$$

$$\mathcal{P} = \mathcal{P}_0 - \frac{3\eta U}{2R}\left(\frac{R}{r}\right)^2\cos\theta \tag{12.35}$$

$\mathcal{P}_0$ is the uniform pressure in the fluid far from the sphere.

12.3.3 Form and Friction Drag

We are now able to calculate the force exerted by the moving fluid on the sphere. In doing this calculation it is helpful to make use of the relation $\mathcal{P} = p + \rho g h$, where p is the pressure and h is the height above an arbitrary datum. We need to work with the pressure and gravity terms separately in order to distinguish the buoyancy term. We take the flow direction to be at an angle α to the direction of gravity, as shown in Fig. 12-4, with $h = 0$ at the center of the sphere. Then $h = r\cos(\theta - \alpha) = r(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$

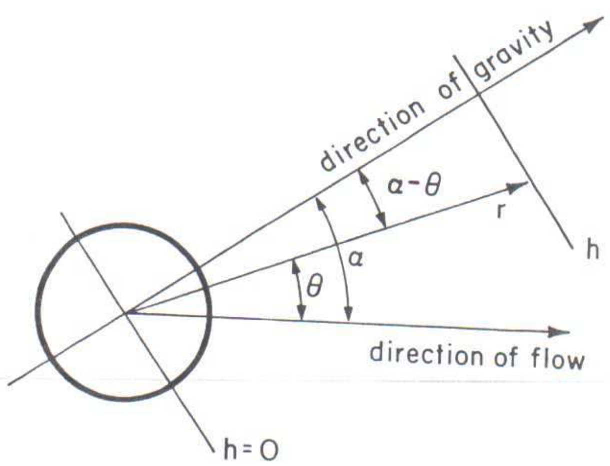

Figure 12-4. Relative orientation of flow and gravity.

and

$$p = \mathcal{P}_0 - \rho g r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) - \frac{3\eta U}{2R}\left(\frac{R}{r}\right)^2 \cos\theta \quad (12.36)$$

We wish to calculate the net force on the sphere in the direction of flow. This is done by multiplying the stress at the surface by the differential surface area, taking the component in the direction of flow, and integrating over the entire surface of the sphere. When the velocity, Eqs. (12.34), is substituted into the stress equations in Table 7-9 we find that the only nonzero term is $\tau_{r\theta}$,

$$\tau_{r\theta} = \eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]$$

At $r = R$, $\tau_{r\theta}$ has the value

$$r = R: \quad \tau_{r\theta} = -\frac{3\eta U}{2R}\sin\theta \quad (12.37)$$

The resolution of the stresses is shown in Fig. 12-5. The component of $\tau_{r\theta}$ in the direction of flow is $-\tau_{r\theta}\sin\theta$, and the component of pressure in the flow direction is $-p\cos\theta$. The differential surface area in spherical coor-

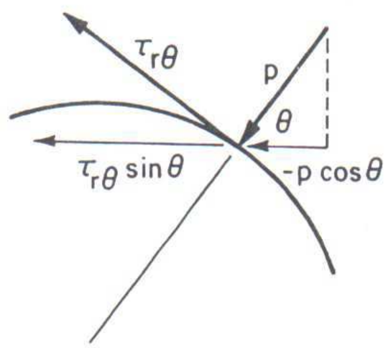

Figure 12-5. Schematic of the stresses acting on the surface of a sphere.

dinates is $R^2\sin\theta\, d\theta\, d\phi$ for $0 \le \theta < \pi$, and $-R^2\sin\theta\, d\theta\, d\phi$ for $\pi \le \theta < 2\pi$. Because of the symmetry about $\theta = 0$, we need only consider the half-plane $0 \le \theta < \pi$, and we obtain the total force by multiplying the result for the half-plane by two. The net force in the flow direction is thus

$$F = 2\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}(-p\cos\theta - \tau_{r\theta}\sin\theta)R^2\sin\theta\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\left[-\mathcal{P}_0 + \rho g R(\cos\alpha\cos\theta + \sin\alpha\sin\theta) + \frac{3\eta U}{2R}\cos\theta\right]$$

$$\cdot R^2\sin\theta\cos\theta\, d\theta\, d\phi + \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\frac{3\eta U}{2R}\sin^2\theta\, R^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{4\pi}{3}\rho g R^3\cos\alpha + 2\pi\eta R U + 4\pi\eta R U \quad (12.38)$$

Note that the result is independent of the pressure at infinity, $\mathcal{P}_0$.

The first term on the right of Eq. (12.38) results from the gravitational force and will be recognized as the *buoyant force*; compare Eq. (4.28). This force will be exerted even in a stationary fluid. The second term ($2\pi\eta RU$) arises from the pressure variation induced by the flow and is often called *form drag*. The final term ($4\pi\eta RU$), which arises from the shear stress at the surface, is often called *friction drag* or *skin friction*. The total drag force induced by flow is

$$F_D = 6\pi\eta RU = 3\pi\eta DU \tag{12.39}$$

This is *Stokes' law*, Eq. (4.4), which we have seen in Fig. 4-1 to be a good representation of experimental data for Re < 1.

*12.3.4 Consistency of Solution

Finally, it is of some interest to estimate the extent of applicability of the solution in order to check for consistency of the approximation. The inertial terms are of order $\rho v_r \partial v_r/\partial r$. With Eq. (12.34a) we can write

$$\text{inertial:}\quad \rho v_r \frac{\partial v_r}{\partial r} = \rho\left\{\left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta\right\} u \cdot \left\{U\left[\frac{3}{2}\frac{R}{r^2} - \frac{3}{2}\frac{R^3}{r^4}\right]\cos\theta\right\} \sim \frac{3\rho U^2 R}{2r^2} \tag{12.40}$$

Here we have retained only the dominant terms, noting that $R \leq r$ and $\cos\theta$ is of order unity. Similarly, the magnitude of the viscous terms is estimated from

$$\text{viscous:}\quad \eta\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right) \sim \frac{3\eta U R^3}{r^5} \tag{12.41}$$

Thus, the ratio of inertial to viscous terms is

$$\frac{\text{inertial}}{\text{viscous}} \sim \frac{3\rho U^2 R/2r^2}{3\eta U R^3/r^5} = \frac{1}{2}\frac{RU\rho}{\eta}\left(\frac{r}{R}\right)^3 = \frac{1}{4}\,\text{Re}\left(\frac{r}{R}\right)^3 \tag{12.42}$$

The ratio of inertial to viscous stresses is small near the sphere as long as Re is small. Sufficiently far from the sphere, however, where $r \gg R$, the inertial terms become comparable to the viscous terms and the creeping flow assumption breaks down. The failure of the assumptions far from the sphere is not important physically, because for $r \gg R$ both the inertial and viscous terms are negligible and the velocity and pressure are close to the uniform values that they would have in the absence of the sphere. Thus, we only need a solution that is valid close to the sphere, and the creeping flow approximation provides this. The inconsistency of the creeping flow solution far from the sphere is of mathematical importance, however, for it restricts the procedures that can be used to construct "corrections" to the creeping flow solution for flows when Re is small but cannot be taken to be zero; specifically, the approach outlined in Sec. 17.2 cannot be used.

12.4 SQUEEZE FILM

The analysis of a squeeze film provides a rather nice example of solution of the creeping flow equations. The squeeze film process is shown schematically in Fig. 12-6. Fluid is contained between two disks of radius R. The upper disk moves toward the lower with a speed V. As the spacing H between the disks decreases, fluid is squeezed out at the edges. The problem is to compute the relationship between the imposed force, the speed and spacing of the plates, and time. Configurations of this type occur in lubrication and in the molding of objects from molten polymers, although the surfaces might be of a more complex shape in practical applications.

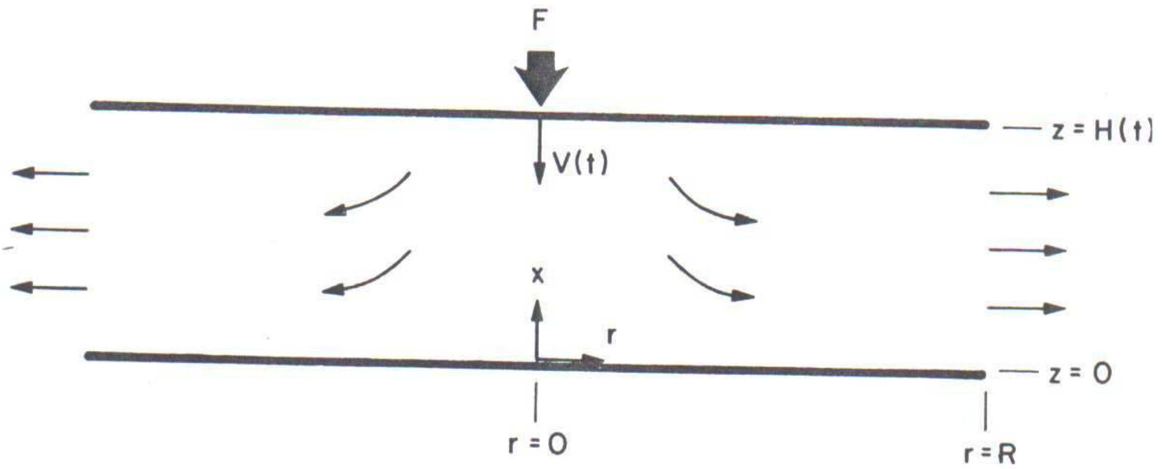

Figure 12-6. Schematic of a squeeze film.

The relevant length scale in the problem is a characteristic spacing between the plates; the initial spacing, H_0, is the maximum value and hence gives the most conservative estimate. The characteristic velocity is the maximum velocity of the upper plate, V_m. The Reynolds number is then

$$\mathrm{Re} = \frac{H_0 V_m \rho}{\eta} \tag{12.43}$$

The only characteristic time in the problem is the time required for the plates to come together, which is of order H_0/V_m. Thus, the characteristic time is constructed from the characteristic length and velocity, and the dimensionless Navier–Stokes equations are as given by Eq. (12.1). In the limit $\mathrm{Re} \rightarrow 0$ we therefore obtain Eq. (12.3b). It is important to note that the time derivative term is neglected, even though we are dealing with a transient flow. *The creeping flow limit for this time-varying flow introduces a*