BOUSSINESQ EQUATIONS FOR STRATIFIED FLOWS

Flows involving fluid motions with variable fluid properties (in particular, variable density) that are due to the existence of non-uniform temperature or solute concentration profiles (in particular, variable salinity) can be described by the usual conservation equations in their most general form.

These equations, however, are very difficult to solve because of their non-linearity. Simplifications are possible in the case of incompressible Newtonian fluid (subject to non-uniform temperature) in the limit of moderate changes of temperature or solute concentration. Under such circumstances, the so-called BOUSSINESQ APPROXIMATION can be invoked.

To explain this approximation, we focus on the case of single-component (single-phase) non-isothermal fluid: for such a fluid, conservation of mass, momentum and thermal energy read as follows

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- $$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\bar{\nabla}\vec{u}\right) = \rho\vec{g} - \bar{\nabla}p + \mu\bar{\nabla}^2\vec{u} + \underbrace{\bar{\nabla}\mu\cdot\left(\overset{\frac{\partial u_i}{\partial x_j}}{\bar{\nabla}\vec{u}} + \overset{\frac{\partial u_j}{\partial x_i}}{\bar{\nabla}\vec{u}^T}\right)}_{\text{NON-LINEAR TERM}}$$

  CONSERVATION OF MOMENTUM

- $$\frac{1}{\rho}\overbrace{\left(\frac{\partial \rho}{\partial t} + \vec{u}\cdot\bar{\nabla}\rho\right)}^{D\rho/Dt} + \bar{\nabla}\vec{u} = 0$$ CONSERVATION OF MASS

- $$\rho C_p\overbrace{\left(\frac{\partial T}{\partial t} + \vec{u}\cdot\bar{\nabla}T\right)}^{DT/Dt} = \underbrace{\overbrace{\bar{\nabla}\left(K\bar{\nabla}T\right)}^{K\bar{\nabla}^2T + \bar{\nabla}K\cdot\bar{\nabla}T}}_{\text{NON-LINEAR TERM}}$$ CONS. OF THERMAL EN

where all fluid properties (ρ, μ, K and C_p) are assumed to depend on spatial position. Hence, $\bar{\nabla}\mu \neq 0$ and $\bar{\nabla}K \neq 0$.

Because of variable fluid properties, if T_0 is the reference fluid temperature then $\rho(T_0) = \rho_0$; $\mu(T_0) = \mu_0$; $K(T_0) = K_0$ and $C_p(T_0) = C_{p,0}$.

To make progress, we note that conservation of momentum in the absence of $\underbrace{\text{shear}}_{\text{NO SPATIAL GRADIENTS OF VELOCITY}}$ or $\underbrace{\text{acceleration}}_{\text{NO TIME CHANGE OF VELOCITY}}$ of the fluid yields:

$$0 = \rho\vec{g} - \bar{\nabla}p_h$$

which is the equation one gets in fluid statics.

If we denote the density of the fluid in the absence of shear or accleration with ρ_o, then:

$$\overline{\nabla} p_h = \rho_o \vec{g}$$

If there is shear or fluid acceleration*, then:

$$p \triangleq p_h + p_d \quad \circledast$$

* With shear or acceleration, $\rho \neq \rho_o$ and therefore $p_{TOT} \neq p_h$

where p_d is the dynamic contribution to the total pressure. The contribution p_d accounts for the fact that the total pressure p acting on the fluid when $\rho \neq \rho_o$ is different from the hydrostatic pressure p_h that would act on the fluid if the density was $\rho = \rho_o$ everywhere.

Because of eq. $\circledast$, the pressure gradient in the conservation equation for momentum is:

$$\overline{\nabla} p = \overline{\nabla} p_h + \overline{\nabla} p_d = \rho_o \vec{g} + \overline{\nabla} p_d$$

and the equation itself becomes:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \overline{\nabla} \vec{u}\right) = (\rho - \rho_o)\vec{g} - \overline{\nabla} p_d + \mu \overline{\nabla} \vec{u} + \overline{\nabla}\mu \cdot \left(\overline{\nabla} \vec{u} + \overline{\nabla} \vec{u}^T\right)$$

We now introduce the Boussinesq approximation, which essentially states that the temperature variations in the fluid are small enough that the fluid properties ρ, μ, κ and c_p can be approximated by their reference values ρ_0, μ_0, κ_0 and $c_{p,0}$ <u>EXCEPT in the body force term</u>, where the approximation $\rho \cong \rho_0$ would mean that the fluid remains motionless.

The Boussinesq approximation can be formalized assuming that the maximum temperature difference $T - T_0$ is small enough to impose the following linear relations:

$$\frac{\rho}{\rho_0} = 1 + \beta (T - T_0) + \cancelto{\text{Neglect}}{E[\mathcal{O}(\Delta T^2)]}$$

[*] $$\frac{\mu}{\mu_0} = 1 + \alpha (T - T_0) + \cancelto{\text{Neglect}}{E[\mathcal{O}(\Delta T^2)]}$$

$$\frac{\kappa}{\kappa_0} = 1 + \gamma (T - T_0) + \cancelto{\text{Neglect}}{E[\mathcal{O}(\Delta T^2)]}$$

$$\frac{c}{c_{p,0}} = 1 + \delta (T - T_0) + \cancelto{\text{Neglect}}{E[\mathcal{O}(\Delta T^2)]}$$

where α, β, γ, δ are coefficients that are generally quite small (for instance, $\beta \simeq 10^{-3}\ {}^\circ C^{-1}$ for liquids).

Upon substitution of eqns. [*] into the conservation equations, under the hypothesis that:

$$\beta \Delta T \, , \; \gamma \Delta T \, , \; \alpha \Delta T \, , \; \delta \Delta T \ll 1$$

being $\Delta T = T - T_0$, one gets:

- $\rho_0 \left(\dfrac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = + \rho_0 \beta \Delta T \vec{g} - \vec{\nabla} p_d + \mu_0 \vec{\nabla}^2 \vec{u}$

- $\vec{\nabla} \vec{u} = 0$

- $\rho_0 C_{p,0} \left(\dfrac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T \right) = K_0 \vec{\nabla}^2 T$

The equations above are known as BOUSSINESQ EQUATIONS, and described natural convection flows.

<u>PROOF</u> FOR MOMENTUM EQUATION: $\begin{cases} \rho \simeq \rho_0 (1 + \beta \Delta T) \\ \mu \simeq \mu_0 (1 + \alpha \Delta T) \end{cases}$

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = (\rho - \rho_0) \vec{g} - \vec{\nabla} p_d + \mu \vec{\nabla} \vec{u} + \vec{\nabla} \mu \cdot \left(\vec{\nabla} \vec{u} + \vec{\nabla} \vec{u}^T \right)$$

$$\rho_0 (1 + \underbrace{\beta \Delta T}_{\simeq 0}) \cdot \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = \rho_0 \beta \Delta T \cdot \vec{g} - \vec{\nabla} p_d + \mu_0 (1 + \underbrace{\alpha \Delta T}_{\simeq 0}) \vec{\nabla}^2 \vec{u}$$

$$+ \underbrace{\underbrace{\vec{\nabla} \left[\mu_0 (1 + \underbrace{\alpha \Delta T}_{\simeq 0}) \right]}_{\vec{\nabla} \mu_0 \simeq 0} \cdot \left(\vec{\nabla} \vec{u} + \vec{\nabla} \vec{u}^T \right)}_{\simeq 0 \, !}$$

$$\rho_0 \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = \rho_0 \beta \Delta T \cdot \vec{g} - \vec{\nabla} p_d + \mu_0 \vec{\nabla}^2 \vec{u} \qquad \text{Q.E.D.}$$

Note that $\beta \triangleq -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p=\text{CONST}}$ is the thermal (or volumetric) expansion coefficient.

Also note that all terms retained in the momentum conservation equation are all of the same order of magnitude ($\mathcal{O}(\Delta T)$, to be specific) whereas all the terms neglected are all $\mathcal{O}(\Delta T)^m$ smaller, where m is some positive exponent.

Although one should expect quantitative deviations from the Boussinesq predictions in situations characterized by large temperature differences (say, larger than 10°C – 20°C), experiments and simulations show that the Boussinesq equations remain qualitatively useful over a considerably larger range of temperature differences. This because the Boussinesq equations preserve the essential property of <u>coupling</u> between the thermal field and the velocity field.

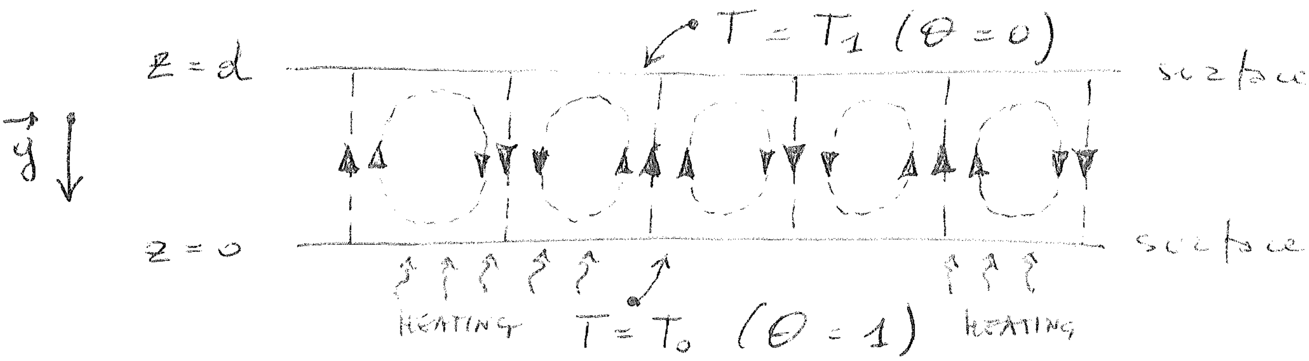
The Boussinesq equations can, of course, be written in non-dimensional form as follows:

$$\tilde{u} = \frac{u}{u_{ref}} \quad ; \quad \tilde{t} = \frac{t \cdot u_{ref}^2}{\nu_0} \quad ; \quad \tilde{x}_i = \frac{x_i\, u_{ref}}{\nu_0}$$

$\nu_0 = \nu_{ref}$!

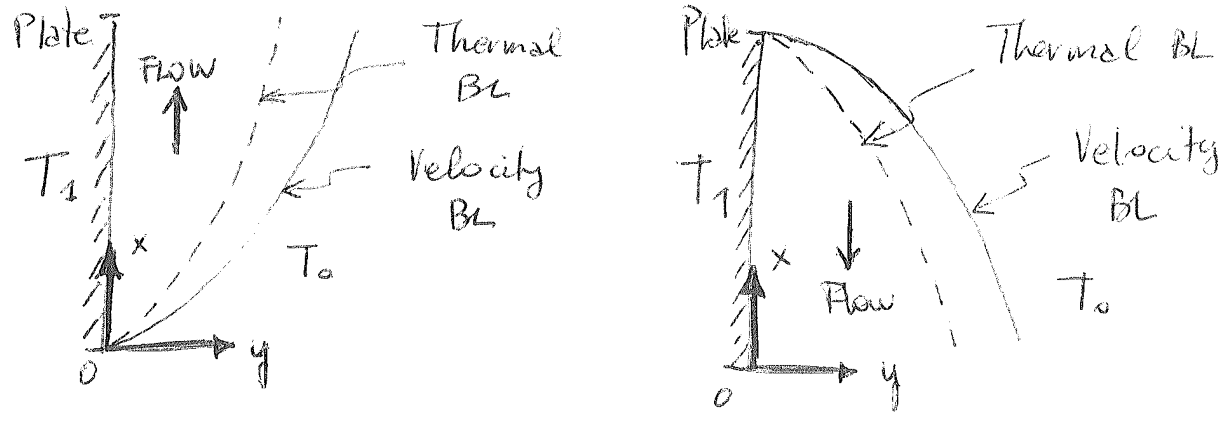
$$\Theta = \frac{T_1 - T}{T_1 - T_0}$$

where T_1 is the fluid temperature at some interface solid boundary. For instance, consider the classic example of natural convection flow known as <u>Rayleigh - Benard problem</u>, in which a stationary "unbounded" layer of fluid between two horizontal plane surfaces is heated from below:



Rayleigh - Benard configuration for buoyancy-driven convection in a horizontal fluid layer that is heated from below: for small temperature difference between the two surfaces, the fluid does not move and heat is exchanged by conduction; for temperature difference larger than a critical value, there is an abrupt and spontaneous transition to heat transfer by convection associated to fluid motions (which give raise to roll cells with motion up and down at alternative cell boundaries)

The same kind of flow is originated if a heated vertical flat plate is considered:



$T_1 > T_0$ $\qquad$ $T_1 < T_0$

Note that the definition of θ yields $\theta = 0$ when $T = T_1$ and $\theta = 1$ when $T = T_0$ (so $0 \leq \theta \leq 1$).

The dimensionless equations read as:

- $$\frac{\partial \tilde{\vec{u}}}{\partial \tilde{t}} + \tilde{\vec{u}} \cdot \tilde{\vec{\nabla}} \tilde{\vec{u}} = -\tilde{\nabla} \tilde{p}_d + \tilde{\nabla}^2 \tilde{\vec{u}} + Gr(1-\theta)\tilde{\vec{g}}$$
- $$\frac{\partial \theta}{\partial \tilde{t}} + \tilde{\vec{u}} \cdot \tilde{\vec{\nabla}} \theta = \frac{1}{Pr} \tilde{\vec{\nabla}}^2 \theta$$
- $$\tilde{\vec{\nabla}} \tilde{\vec{u}} = 0$$

NOTE: $Re = \frac{u_{ref} \cdot l_{ref}}{\nu_0} \overset{!}{=} 1$ in this case !!

where $\tilde{\vec{\nabla}} = \frac{\partial \cdot}{\partial \tilde{x}_i}$ and:

$$\boxed{Gr \triangleq \beta (T_1 - T_0) g \cdot \frac{\nu_0}{u_{ref}^3} \overset{!}{=} \frac{\beta (T_1 - T_0) g\, l_{ref}^3}{\nu_0^2}}$$

$\boxed{l_{ref} = \frac{\nu_0}{u_{ref}}}$

GRASHOF NUMBER

Physical meaning:

$$Gr = \frac{\text{buoyancy forces}}{\text{viscous forces}}$$

Recalling that $Ri = \dfrac{\text{buoyancy forces}}{\text{inertial forces}}$, it follows that the ratio Gr/Ri has the same physical meaning of a Reynolds number:

$$\frac{Gr}{Ri} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

Indeed:

$$\frac{Gr}{Ri} = \frac{\beta \cdot \Delta T \cdot \cancel{g} \cdot l_{ref}^{\cancel{3}\,2}}{\nu_0^2} \cdot \frac{1}{\dfrac{\Delta \rho}{\rho_0} \cdot \dfrac{\cancel{g}\, l_{ref}}{U_{ref}^2}}$$

$$\beta = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_{p=const} \simeq -\frac{1}{\rho_0}\frac{\Delta \rho}{\Delta T}$$

$$= -\frac{1}{\rho_0}\frac{\Delta \rho}{\Delta T} \cdot \frac{\Delta T \, l_{ref}^2}{\dfrac{\Delta \rho}{\rho_0} \cdot \dfrac{1}{U_{ref}^2}} \cdot \frac{1}{\nu_0^2} = \frac{U_{ref}^2 \cdot l_{ref}^2}{\nu_0^2}$$

$$= Re^2$$

Hence: $\boxed{Gr \cdot Ri^{-1} = Re^2}$ Q.E.D.

where it is assumed that ρ_0 and ν_0 are the density and the kinematic viscosity of the fluid at the reference temperature T_0 whereas l_{ref} and u_{ref} are the reference length and velocity of the convective fluid flow.

Let us now write the dimensionless Boussinesq equations for the Rayleigh-Bénard (buoyancy-driven) convection in a horizontal fluid layer heated from below:

$$[1] \quad \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{\partial p_d}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} + \cancel{Gr(1-\theta)}\, g_x^{\,=0} \quad (g_x = 0)$$

$$[2] \quad \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} = -\frac{\partial p_d}{\partial y} + \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} + \cancel{Gr(1-\theta)}\, g_y^{\,=0} \quad (g_y = 0)$$

$$[3] \quad \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p_d}{\partial z} +$$

$$+ \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z} + Gr(1-\theta)\, g_z$$

$$[4] \quad \frac{\partial \theta}{\partial t} + u_x \frac{\partial \theta}{\partial x} + u_y \frac{\partial \theta}{\partial y} + u_z \frac{\partial \theta}{\partial z} = \frac{1}{Pr}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right)$$

$$[5] \quad \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

We now recall that, in the Rayleigh-Bénard problem, there is no fluid motion below a critical value of the temperature difference whereas (convective) fluid motions are generated above such critical value.

Therefore, if we are <u>below the critical temperature difference</u>, then $u_x = u_y = u_z = 0$ and the Boussinesq equations become:

$$[1]' \quad 0 = -\frac{\partial P_d}{\partial x}$$

$$[2]' \quad 0 = -\frac{\partial P_d}{\partial y}$$

$$[3]' \quad 0 = -\frac{\partial P_d}{\partial z} + Gr(1-\theta)\, g_z$$

$$[4]' \quad 0 = \frac{\partial^2 \theta}{\partial z^2}$$

(due to steady state - $\frac{\partial \cdot}{\partial t} = 0$ - and 1D temp field)

whereas continuity reduces to $0 = 0$

This yields:

$$\boxed{dp_d = -\,Gr\,(1-\theta)\,g_z\,dz}\;\Big|_{g_z=-g}$$

$$\theta = C_1 \cdot z + C_2$$

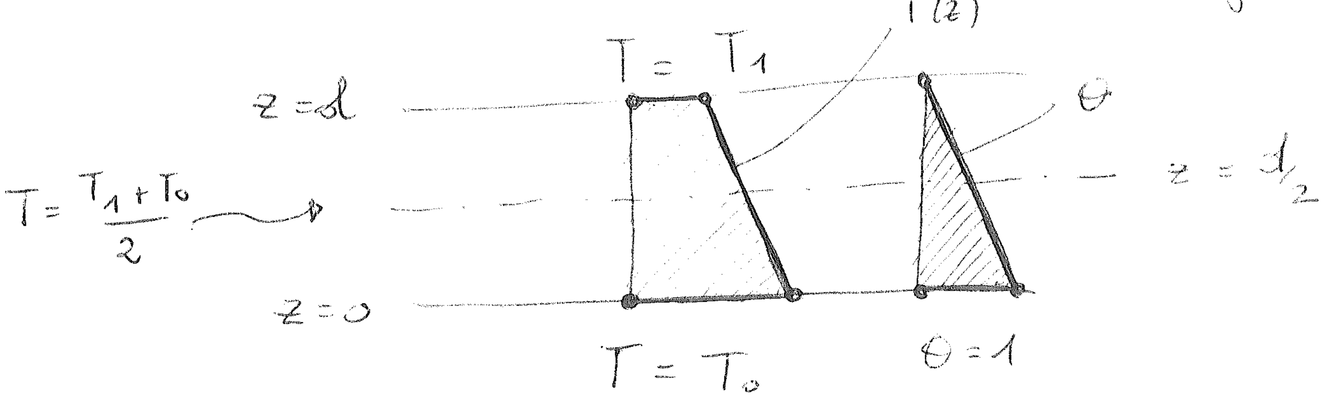
where C_1 and C_2 are integration constants that can be computed from the boundary conditions.

$$\left.\begin{aligned} \theta(z=0) = 1 \quad &\Rightarrow \quad C_2 = 1 \\ \theta(z=d) = 0 \quad &\Rightarrow \quad C_1 = -1/d \end{aligned}\right\} \boxed{\theta = \left(1 - \frac{z}{d}\right)}$$

Recalling that $\theta \triangleq \dfrac{T_1 - T}{T_1 - T_0}$, one finds:

$$\boxed{T(z) = T_1 - (T_1 - T_0)\left(1 - \frac{z}{d}\right)}$$

Temperature profile in the absence of fluid motion



If we are above the critical temperature difference, then fluid motions start to

occur due to the onset of convection.
At the beginning of the process (namely soon after the onset of convection), fluid motions are equivalent to perturbations of the velocity field. In other words, convection induces small perturbations to the fluid velocity, which becomes different from zero within the fluid layer.

As long as these perturbations are small, one can write:

$$\left.\begin{aligned} u_x &= \delta\,\tilde{u}_x \\ u_y &= \delta\,\tilde{u}_y \\ u_z &= \delta\,\tilde{u}_z \end{aligned}\right\} \text{velocity perturbations}$$

$$\theta = \overbrace{\left(1 - \frac{z}{d}\right)}^{\bar{\theta}} + \delta\,\tilde{\theta} \qquad \text{Temperature perturb.}$$

$$p_d = \overbrace{Gr\,(1-\theta)\,g \cdot z}^{\bar{P}_d} + \delta\,\tilde{p}_d \qquad \text{Pressure perturb.}$$

where δ represents the amplitude of the perturbation, and the Boussinesq equations become:

$$[1]'' \quad \frac{\partial(\delta \tilde{u}_x)}{\partial t} + \cancelto{\simeq 0}{\delta \tilde{u}_x \frac{\partial(\delta \tilde{u}_x)}{\partial x}} + \cancelto{\simeq 0}{\delta \tilde{u}_y \frac{\partial(\delta \tilde{u}_x)}{\partial y}} +$$

$$+ \cancelto{\simeq 0}{\delta \tilde{u}_z \frac{\partial(\delta \tilde{u}_x)}{\partial z}} = - \cancelto{= 0 \text{ since } \bar{P}_d = \bar{P}_d(z)!}{\frac{\partial \bar{P}_d}{\partial x}} - \frac{\partial(\delta \tilde{P}_d)}{\partial x} +$$

$$+ \frac{\partial^2(\delta \tilde{u}_x)}{\partial x^2} + \frac{\partial^2(\delta \tilde{u}_x)}{\partial y^2} + \frac{\partial^2(\delta \tilde{u}_x)}{\partial z^2}$$

with $\delta \tilde{u}_x \frac{\partial(\delta \tilde{u}_x)}{\partial x} \simeq O(\delta^2)$ hence negligible

$\delta \tilde{u}_y \frac{\partial(\delta \tilde{u}_x)}{\partial y} \simeq O(\delta^2)$ hence negligible

$\delta \tilde{u}_z \frac{\partial(\delta \tilde{u}_x)}{\partial z} \simeq O(\delta^2)$ hence negligible

compared to the other terms, which are all $O(\delta)$. In a more compact form:

$$\boxed{\frac{\partial(\delta \tilde{u}_x)}{\partial t} = - \frac{\partial(\delta \tilde{P}_d)}{\partial x} + \underbrace{\frac{\partial^2(\delta \tilde{u}_x)}{\partial x^2} + \frac{\partial^2(\delta \tilde{u}_x)}{\partial y^2} + \frac{\partial^2(\delta \tilde{u}_x)}{\partial z^2}}_{\tilde{\nabla}^2(\delta \tilde{u}_x)}}$$

Since a similar equation can be obtained for $\delta \tilde{u}_y$ and $\delta \tilde{u}_z$, the full derivation is omitted and only the final form of the equations is written:

$$[2]'' \quad \boxed{\frac{\partial(\delta\tilde{u}_y)}{\partial t} = -\frac{\partial(\delta\tilde{p}_d)}{\partial y} + \bar{\nabla}^2(\delta\tilde{u}_y)}$$

$$[3]'' \quad \boxed{\frac{\partial(\delta\tilde{u}_z)}{\partial t} = -\frac{\partial(\delta\tilde{p}_d)}{\partial z} + \bar{\nabla}^2(\delta\tilde{u}_z) - Gr\cdot\delta\tilde{\theta}}$$

$$[4]'' \quad \cancelto{=0}{\frac{\partial\bar{\theta}}{\partial t}} + \frac{\partial(\delta\tilde{\theta})}{\partial t} + \cancelto{=0}{\delta\tilde{u}_x\frac{\partial\bar{\theta}}{\partial x}} + \cancelto{=0}{\delta\tilde{u}_x\frac{\partial(\delta\tilde{\theta})}{\partial x}}$$

$$\bar{\theta} = \bar{\theta}(z)$$

$$+ \cancelto{=0}{\delta\tilde{u}_y\frac{\partial\bar{\theta}}{\partial y}} + \cancelto{=0}{\delta\tilde{u}_y\frac{\partial\tilde{\theta}}{\partial y}} + \delta\tilde{u}_z\frac{\partial\bar{\theta}}{\partial z} +$$

$$+ \cancelto{=0}{\delta\tilde{u}_z\frac{\partial(\delta\tilde{\theta})}{\partial z}} = \frac{1}{Pr}\left(\cancelto{=0}{\frac{\partial^2\bar{\theta}}{\partial x^2}} + \frac{\partial^2(\delta\tilde{\theta})}{\partial x^2} + \cancelto{=0}{\frac{\partial^2\bar{\theta}}{\partial y^2}} + \right.$$

$$\left. + \frac{\partial^2(\delta\tilde{\theta})}{\partial y^2} + \cancelto{=0}{\frac{\partial^2\bar{\theta}}{\partial z^2}} + \frac{\partial^2(\delta\tilde{\theta})}{\partial z^2}\right)$$

$$\bar{\theta} = 1 - \frac{z}{d} \;!$$

This yields: $\frac{\partial(\delta\tilde{\theta})}{\partial t} + \delta\tilde{u}_z\frac{\partial\bar{\theta}}{\partial z} = \frac{1}{Pr}\bar{\nabla}^2(\delta\tilde{\theta})$

with $\frac{\partial\bar{\theta}}{\partial z} = -\frac{1}{d}$: $\boxed{\frac{\partial(\delta\tilde{\theta})}{\partial t} - \frac{1}{d}\delta\tilde{u}_z = \frac{1}{Pr}\bar{\nabla}^2(\delta\tilde{\theta})}$

$$[5]'' \quad \frac{\partial(\delta\tilde{u}_x)}{\partial x} + \frac{\partial(\delta\tilde{u}_y)}{\partial y} + \frac{\partial(\delta\tilde{u}_z)}{\partial z} = 0$$

The bottom line of this derivation is that the Boussinesq equations can be <u>linearized</u> if convection produces small perturbations in the velocity, temperature and pressure fields. When non-linear terms can be neglected, the problem and its mathematical tractability are greatly simplified.

Note that an alternative form of these linearized equations can be obtained if the temperature perturbation is expressed as $\delta\hat{\theta} = \delta\tilde{\theta}/Pr$ (this rescaling has been used more often for historical reasons). Replacing $\delta\tilde{\theta}$ with $\delta\hat{\theta} \cdot Pr$ yields:

$$[3]''' \quad \frac{\partial(\delta\tilde{u}_z)}{\partial t} = -\frac{\partial(\delta\tilde{p}_d)}{\partial z} + \bar{\nabla}^2(\delta\tilde{u}_z) - Gr \cdot Pr\,\delta\hat{\theta}$$

$$[4]''' \quad Pr\,\frac{\partial(\delta\hat{\theta})}{\partial t} - \frac{1}{1}\,\delta\tilde{u}_z = \bar{\nabla}^2(\delta\hat{\theta})$$

where :

$$\boxed{Gr \cdot Pr = Ra}$$ RAYLEIGH NUMBER

The introduction of the Rayleigh number justifies the rescaling $\delta\hat{\theta} = \delta\tilde{\theta}/Pr$. When Ra is below a critical value then heat transfer within a heated fluid layer occurs primarily by conduction (and convection is less important or even negligible). When Ra exceeds the critical value then heat transfer within the heated fluid layer occurs primarily by convection (and conduction is less important or even negligible). Hence Ra can be used as dimensionless parameter to characterize the critical condition required to trigger convection in a Rayleigh - Bénard problem.

Lord Rayleigh himself showed that the critical value of Ra required to generate convective fluid motions in a layer bounded by two free surfaces (one heated at temp. T_0 and one cooled at temperature $T_1 < T_0$) is $R_c = \frac{27\pi^4}{4} \simeq 657$.

If the heated fluid layer is confined by a solid boundary at the bottom and by a free surface at the top, then the critical value becomes $R_c \simeq 1100$.