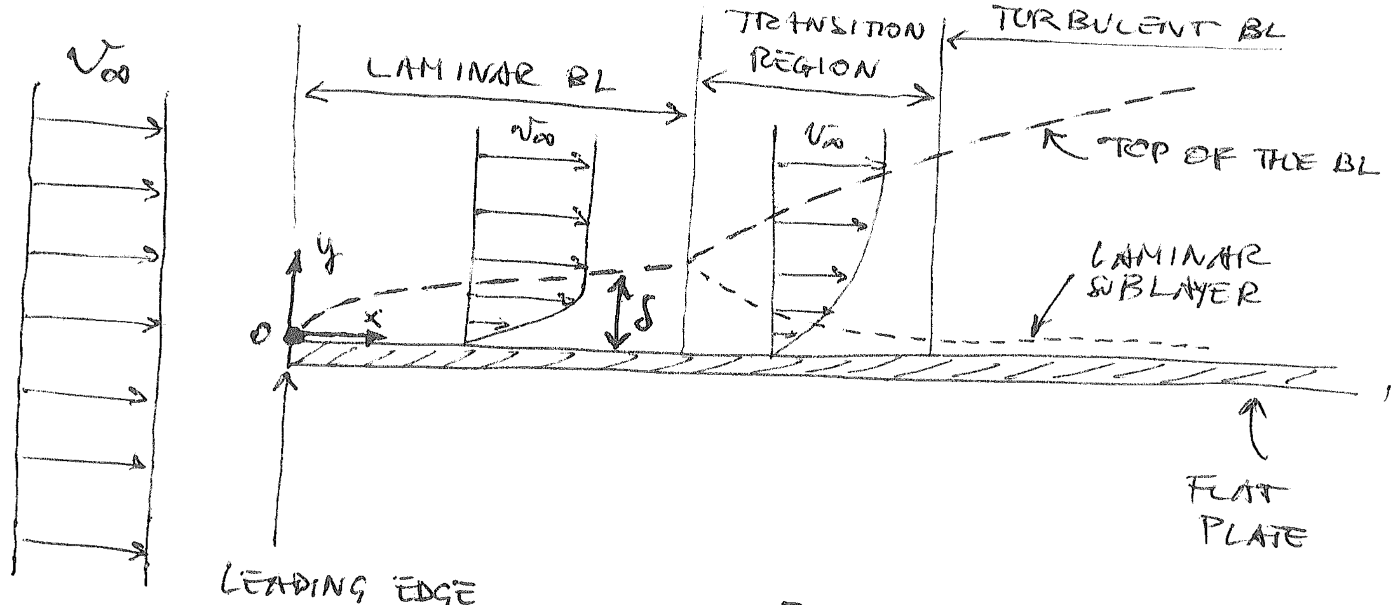
BOUNDARY LAYER

The concept of boundary layer (BL) was introduced in the beginning of the 20th century by Ludwig Prandtl and Karl Pohlhausen to study flow phenomena and fluid interaction with a solid boundary.

In environmental flows, only a minor portion of the flow domain is characterized by large velocity gradients, large shear stresses and viscous dissipation (which are the main macroscopic features of fluid behavior in the BL): Other portions of the domain, located outside the BL and further away from the solid boundary, are not "affected" by viscous dissipation and are not characterized by strong, persistent velocity gradients (so that potential flow conditions can be applied).

Indeed, with fluids like air or water (characterized by relatively small viscosity) the spatial extent of the BL is limited to small distances from the solid boundary. Only within such small distance, viscous forces and friction play a role and must be accounted for. Outside, inertial forces become predominant compared to viscous forces and the flow is virtually unaffected by friction. Note that the extent of the BL decrease as the Reynolds number of the flow increases.

The simplest instance of BL is associated with the development of the flow field over an infinite horizontal plate (BLASIUS PROBLEM)



BL DEVELOPMENT OVER A FLAT PLATE

Development of the BL begins at the plate's leading edge ($x=0$): downstream of the leading edge, the flow inside the BL is laminar, for larger distances there is a transition region (in which flow instabilities start to appear so that the flow cannot be purely laminar), followed by a region in which the flow is turbulent over a very thin sublayer where the flow is laminar (very close to the plate, viscosity still dominates over inertia even when the flow in the remaining portion of the BL has become turbulent).

EQUATIONS OF THE BL (IN 2D):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

Some approximations can be made:

1) $\frac{\partial v_x}{\partial y} \gg \frac{\partial v_x}{\partial x}$ (v_x changes much more along the vertical direction, y, than along the horizontal direction, x, in the BL)

From this condition, it follows : $\frac{\partial^2 v_x}{\partial y^2} \gg \frac{\partial^2 v_x}{\partial x^2}$

2) $v_x \gg v_y$ but $\frac{\partial v_x}{\partial x} \ll \frac{\partial v_x}{\partial y}$. Therefore :

$$v_x \frac{\partial v_x}{\partial x} \simeq v_y \frac{\partial v_x}{\partial y}$$

3) $$\frac{\partial P}{\partial x} = \frac{\partial p}{\partial x} + \rho g \underbrace{\frac{\partial h}{\partial x}}_{\simeq 0} \simeq \frac{\partial p}{\partial x}$$

From Bernoulli : $p + \frac{1}{2}\rho v^2 + \rho g h = \text{CONST.}$

Outside the BL $\rightarrow$ $p + \frac{1}{2}\rho v_\infty^2 + \rho g h = \text{CONST.}$

$$\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho v_\infty^2\right) + \underbrace{\cancel{\frac{\partial}{\partial x}(\rho g h)}}_{=0} = 0$$

$$\frac{\partial p}{\partial x} = -\frac{1}{2}\rho \frac{\partial v_\infty^2}{\partial x}$$

$= 0$ if $v_\infty = \text{CONST.}$

$= -\rho v_\infty \frac{d v_\infty}{dx}$ if $v_\infty = v_\infty(x)$

Therefore, in the general case $v_\infty = v_\infty(x)$, in which the incoming fluid velocity can change along the horizontal direction due to flow acceleration or deceleration, one gets:

$$\boxed{\frac{\partial p}{\partial x} = -\rho\, v_\infty \frac{d v_\infty}{dx}} \quad [1]$$

Clearly $\partial p/\partial x = 0$ if v_∞ = constant in time and uniform in space.

Based on the considerations made above, the equations for the B.L. can be written as follows (in 2D):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{CONT.}$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) \cong -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2} \qquad NS_x$$

$$0 \cong -\frac{\partial p}{\partial y} \qquad NS_y$$

where NS_y has been approximated considering that its non-dimensional form is:

$$\underbrace{\left(\frac{\delta}{L}\right)^2}_{\ll 1 \text{ since } \delta \ll L}\left(\frac{\partial \tilde{v}_y}{\partial \tilde{t}} + \tilde{v}_x \frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_y}{\partial \tilde{y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \underbrace{\left(\frac{\delta}{L}\right)^4}_{\ll 1 \text{ since } \delta \ll L}\left(\frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}\right)$$

where δ is the B.L. thickness.

Note that the condition $\partial p/\partial y = 0$ implies that in 2D: $P = P(x)$ only!

If P is independent of y, then the expression [1] can be used to express the equivalent-pressure gradient for any coordinate y, namely the expression is valid also inside the B.L. (not only outside). Therefore the governing equations become:

$$\begin{cases} \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0 \\ \rho\left(\dfrac{\partial v_x}{\partial t} + v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y}\right) = \rho v_\infty \dfrac{d v_\infty}{dx} + \mu \dfrac{\partial^2 v_x}{\partial y^2} \quad [2] \\ 0 = -\dfrac{\partial P}{\partial y} \end{cases}$$

Solution of this system of Partial Differential Equations (PDE) would provide the fluid velocity components, v_x and v_y, inside the B.L. However, equation [2] cannot be solved analitically, just numerically.

To solve the equation, the following dimensionless variable can be introduced:

→

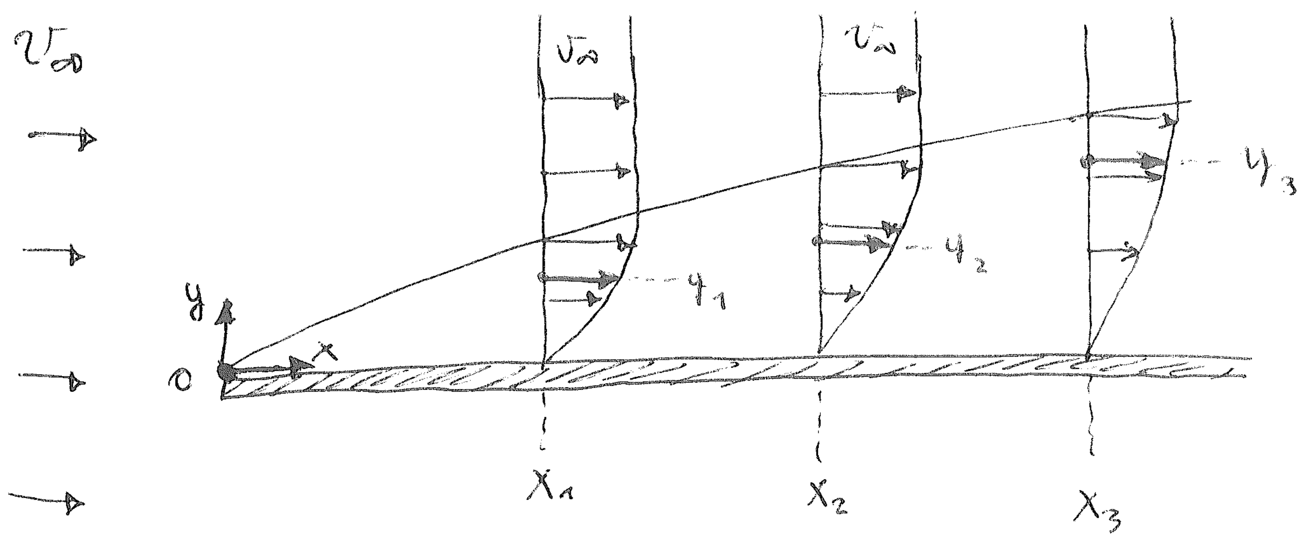
$$\boxed{\eta \triangleq \frac{y}{\delta}}$$ SIMILARITY VARIABLE

Through this variable, it is always possible to express the fluid velocity in the horizontal direction as:

$$\boxed{v_x = v_\infty \cdot \phi(\eta)}$$ SIMILARITY PROFILE

where $\phi(\eta)$ is a function of η (and of η only) that changes depending on the type of B.L.

I) B.L. ON A FLAT PLATE (BLASIUS PROBLEM)



This B.L. develops in space (along the plate) but does not depend on time, so the problem is stationary. The fluid velocity vector has two non-zero components:

$$v_x = v_x(x,y) \neq 0$$

$$v_y = v_y(x,y) \neq 0 \rightsquigarrow v_y \ll v_x$$

$$v_z = 0$$

Note that it is always possible to relate the velocity profiles at different locations along the plate to each other. In the schematic, for instance, the coordinates $(x_1; y_1)$, $(x_2; y_2)$ and $(x_3; y_3)$ can be chosen such that:

$$v_x(x_1; y_1) = v_x(x_2; y_2) = v_x(x_3; y_3) \quad (*)$$

The similarity variable is defined as:

$$\eta \triangleq y/\delta(x)$$

so one can rewrite $(*)$ as: $v_x(\eta_1) = v_x(\eta_2) = v_x(\eta_3)$.
In addition, we assume $v_\infty = \text{CONST.}$ so $\partial P/\partial x = 0$.

Let us now rewrite continuity and NS equations using $\eta = y/\delta(x)$ and $v_x = v_\infty \cdot \phi(\eta)$:

CONTINUITY:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(v_\infty \phi(\eta)\right) + \frac{\partial v_y}{\partial y} = 0$$

$$v_\infty \cdot \frac{\partial \phi(\eta)}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Now:

$$\boxed{\frac{\partial \cdot}{\partial x}} = \frac{\partial \cdot}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \boxed{-\frac{y}{\delta^2(x)} \cdot \frac{d\delta(x)}{dx} \cdot \frac{\partial \cdot}{\partial \eta}} \quad [3]$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{\delta(x)}\right) = y \, \frac{\partial \delta^{-1}(x)}{\partial x} =$$

$$= -\frac{y}{\delta^2(x)} \frac{d\delta(x)}{dx} = -\frac{\eta}{\delta(x)} \cdot \frac{d\delta(x)}{dx}$$

$$\boxed{\frac{\partial \cdot}{\partial y}} = \frac{\partial \cdot}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \boxed{\frac{1}{\delta(x)} \cdot \frac{\partial \cdot}{\partial \eta}} \quad [4]$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta(x)}$$

Replacing eqns. [3] and [4] into continuity one gets:

$$-v_\infty \frac{y}{\delta^2(x)} \cdot \frac{d\delta}{dx} \cdot \frac{\partial \phi(\eta)}{\partial \eta} + \frac{1}{\delta(x)} \cdot \frac{\partial v_y}{\partial \eta} = 0$$

Setting $\partial\phi(\eta)/\partial\eta \equiv d\phi(\eta)/d\eta = \phi'$:

$$-v_\infty \cdot \eta \cdot \frac{1}{\cancel{\delta(x)}} \cdot \frac{d\delta(x)}{dx} \cdot \phi' + \frac{1}{\cancel{\delta(x)}} \cdot \frac{\partial v_y}{\partial \eta} = 0$$

$$\frac{\partial v_y}{\partial \eta} = v_\infty \eta \frac{d\delta}{dx} \phi'$$

Integrate in η

$$v_y = v_\infty \frac{d\delta}{dx} \underbrace{\int_0^\eta \eta \phi' d\eta}_{}$$

$$= -\int_0^\eta \phi \underbrace{\eta'}_{} d\eta + \eta \cdot \phi$$

$$\eta' = \frac{\partial \eta}{\partial \eta} = 1$$

$$= \left[\eta \cdot \phi - \int_0^\eta \phi \, d\eta\right] v_\infty \frac{d\delta}{dx}$$

Let us now set : $f(\eta) \triangleq \int_0^\eta \phi \, d\eta$

We find :

$$f'(\eta) = \frac{df}{d\eta} = \phi$$

Therefore : $\boxed{v_y = (\eta \cdot f' - f) v_\infty \frac{d\delta}{dx}}$ [5] ; $\boxed{v_x = v_\infty \cdot f'(\eta)}$ [6]

In summary, we have been able to rewrite the fluid velocity components as a function of the similarity variable only!

Using the above expressions, we can now rewrite

NS_x:

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \qquad \left(\frac{\partial p}{\partial x} = 0 \, !\right)$$

$$\underbrace{\cancel{\frac{\partial (v_\infty \cdot f')}{\partial t}}}_{=0 \text{ since } f' = f'(\eta) \text{ but } f' \neq f'(t)!} + v_\infty f' \cdot \underbrace{\frac{\partial}{\partial x}(v_\infty \cdot f')}_{v_\infty \frac{\partial f'}{\partial x} = v_\infty \left[-\frac{\eta}{\delta(x)} \cdot \frac{d\delta}{dx} \frac{\partial f'}{\partial \eta}\right] = -v_\infty \frac{\eta}{\delta(x)} \cdot \frac{d\delta}{dx} f''} + v_\infty(\eta \cdot f' - f) \underbrace{\frac{\partial (v_\infty \cdot f')}{\partial y}}_{v_\infty \partial f' / \partial y = v_\infty / \delta(x) \cdot f''} \cdot \frac{d\delta}{dx}$$

$$= \nu \underbrace{\frac{\partial}{\partial y}\left[\underbrace{\frac{\partial (v_\infty \cdot f')}{\partial y}}_{\frac{v_\infty}{\delta(x)} \cdot f''}\right]}_{\frac{v_\infty}{\delta^2(x)} \cdot f'''}$$

$$-v_\infty^2 \frac{\eta}{\delta(x)} \cdot \frac{d\delta}{dx} \cdot f' \cdot f'' + (\eta \cdot f' - f) \frac{v_\infty^2}{\delta(x)} \frac{d\delta}{dx} f'' = \frac{\nu v_\infty}{\delta^2(x)} f'''$$

$$-\cancel{v_\infty^2 \frac{\eta}{\delta(x)} \frac{d\delta}{dx} f' \cdot f''} + \cancel{v_\infty^2 \frac{\eta}{\delta(x)} \frac{d\delta}{dx} f' \cdot f''} - \frac{v_\infty^2}{\delta(x)} \frac{d\delta}{dx} f \cdot f'' =$$

$$= \frac{\nu v_\infty}{\delta^2(x)} f'''$$

$$\boxed{f''' + \frac{v_\infty \delta(x)}{\nu} \cdot \frac{d\delta}{dx} f \cdot f'' = 0} \quad [7]$$

This equation is just the NS_x rewritten in terms of the unknown function $f(\eta)$.
By solving this equation, we can find $f(\eta)$ and, in turn, both v_x and v_y through eqns. [5] and [6] at page 10, respectively.

Note that:

$$\frac{v_\infty \delta(x)}{\nu} \cdot \frac{d\delta}{dx} = \text{CONST.} = C$$

Upon separation of variables:

$$\int_{\delta(x=0)=0}^{\delta(x)} \delta(x)\, d\delta = C \frac{\nu}{v_\infty} \int_0^x dx \Rightarrow \boxed{\delta(x) = \sqrt{\frac{2C\nu}{v_\infty} \cdot x}} \quad [8]$$

This equation provides an expression for the

Blasius B.L. thickness.

Note that, regardless of the specific value of C, eq. [8] yields:

$$\boxed{\delta(x) \propto x^{1/2}}$$

in the laminar region of the B.L.

It can also be observed that the solution of Eq. (7) does not depend on the specific value chosen for the constant C. In the literature, the most common values are:

$$C = \frac{1}{2} \Rightarrow \delta(x) = \sqrt{\frac{\nu \cdot x}{v_\infty}} \Rightarrow f \cdot f'' + 2 f''' = 0$$

- $$\frac{d\delta}{dx} = \frac{1}{2}\sqrt{\frac{\nu}{v_\infty \cdot x}}$$
- $$\frac{v_\infty}{\nu}\delta(x)\frac{d\delta}{dx} = \frac{v_\infty}{\nu}\sqrt{\frac{\nu \cdot x}{v_\infty}} \cdot \frac{1}{2}\sqrt{\frac{\nu}{v_\infty \cdot x}} = \frac{1}{2}$$

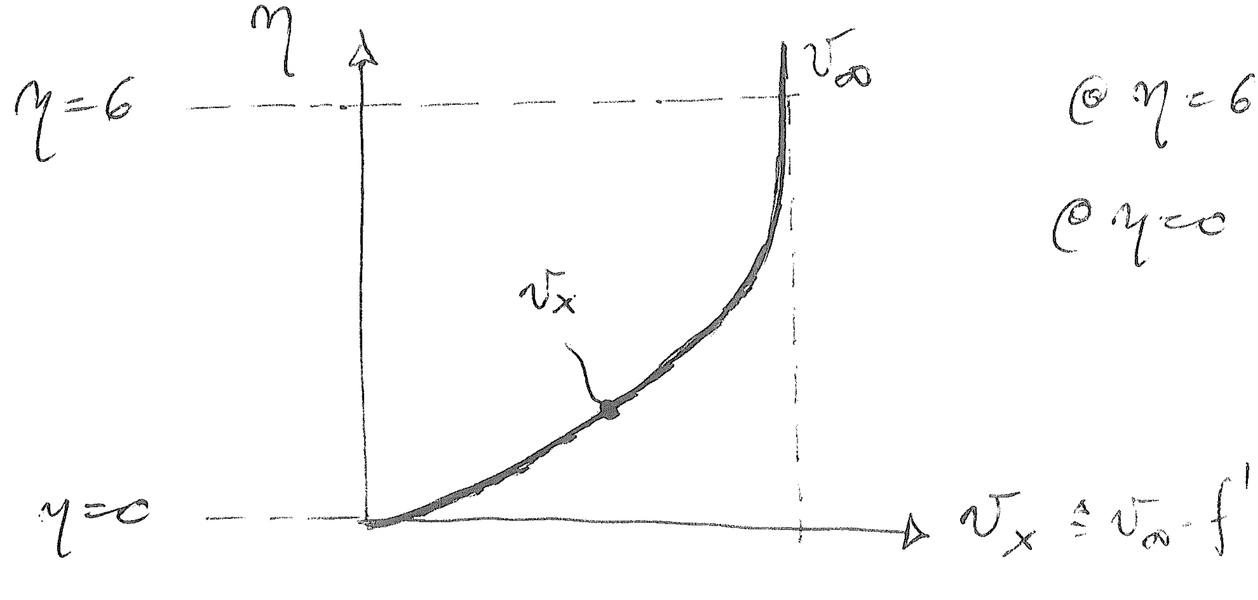
$$C = 1 \Rightarrow \delta(x) = \sqrt{\frac{2\nu x}{v_\infty}} \Rightarrow f \cdot f'' + f''' = 0$$

$$d\delta/dx = \frac{1}{2}\sqrt{\frac{2\nu}{v_\infty x}} \; ; \; \frac{v_\infty}{\nu}\delta(x)\frac{d\delta}{dx} = \frac{v_\infty}{\nu}\sqrt{\frac{2\nu x}{v_\infty}} \cdot \frac{1}{2}\sqrt{\frac{2\nu}{v_\infty x}} = 1$$

Let us solve for $f \cdot f'' + 2 f''' = 0$ with $\delta(x) = \sqrt{\frac{\nu \cdot x}{v_\infty}}$. This equation can only be solved numerically using the following boundary conditions:

- $v_x(y=0) = 0 \xrightarrow{v_x = v_\infty \cdot f'} f'(\eta = 0) = 0$
- $v_y(y=0) = 0 \xrightarrow{v_y = \dots} \underbrace{\eta \cdot f'(\eta=0)}_{=0} - f(\eta = 0) = 0$
- $v_x(y \to +\infty) = v_\infty \Longrightarrow f'(\eta \to \infty) = 1$

$$[9] \quad \begin{cases} f \cdot f'' + 2 f''' = 0 \\ f'(\eta = 0) = 0 \\ f(\eta = 0) = 0 \\ f'(\eta \to \infty) = 1 \end{cases} \xrightarrow{\text{SOLUTION}}$$



@ $\eta = 6$: $v_x \simeq v_\infty$

@ $\eta = 0$: WALL!

Through this plot the fluid velocity at any point $P(x, y)$ inside the B.L. can be found according to this procedure:

1. Select P, namely choose the values of x_P, y_P
2. From x_P compute $\delta_P(x) = \sqrt{\frac{\nu \cdot x_P}{v_\infty}}$
3. From $\delta_P(x)$ and y_P compute $\eta_P = y_P / \delta_P(x)$
4. From η_P enter the plot and find $v_{x,P}$

Once the velocity is known, one can compute the shear stress at the wall and the vorticity:

$$\tau_w = \tau_{xy}\Big|_{y=0} = \mu \frac{\partial v_x}{\partial y}\Big|_{y=0} = \mu \frac{\partial (v_\infty \cdot f')}{\partial y}\Big|_{y=0}$$

$$= \frac{\mu v_\infty}{\delta(x)} \cdot \frac{\partial f'}{\partial \eta}\Big|_{y=0} = \frac{\mu v_\infty}{\delta(x)} \cdot f''\Big|_{y=0}$$

However, $y=0 \Rightarrow \eta=0$ and numerical solution of eqn. [9] yields $f''(\eta=0) = 0,332$.

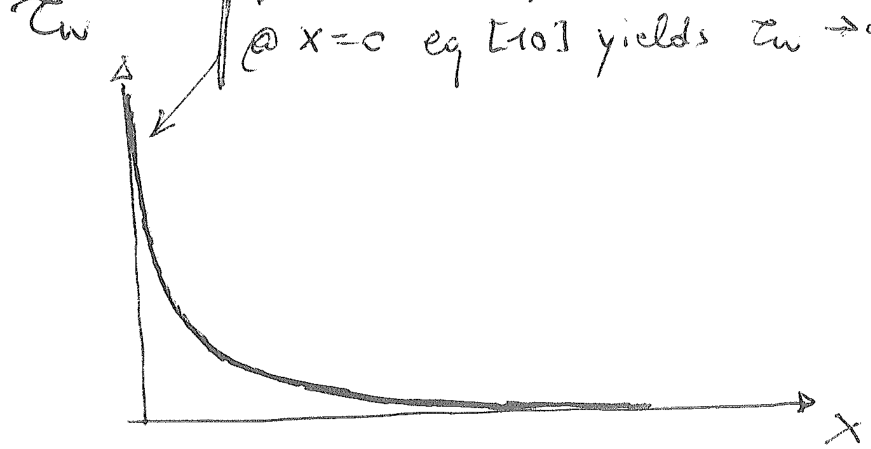
It is found that:

$$\boxed{\tau_w = 0{,}332\,\frac{\mu \cdot v_\infty}{\delta(x)}}$$

$\delta(x) = \sqrt{\frac{\nu x}{v_\infty}} \rightsquigarrow$

$$= \boxed{\underbrace{0{,}332\,\mu\sqrt{\frac{v_\infty^3}{\nu}}}_{\text{CONSTANT}} \cdot x^{-1/2}} \qquad [10]$$

In the laminar region of the flow: $\tau_w \propto x^{-1/2}$
Therefore τ_w decreases as x increases (along the plate):



EXAMPLE OF CALCULATION OF $\delta(x)$ AND τ_w:

$v_\infty = 1\ m/s$

$\nu = 1.57 \cdot 10^{-5}\ m^2/s$ (AIR) $\Rightarrow \begin{cases} \delta(x) = 1{,}253\ cm \\ \tau_w = 5{,}3 \cdot 10^{-4}\ N/m^2 \end{cases}$

$x = 10\ m$

With water ($\nu = 10^{-6}$): $\delta(x = 10m) \simeq 0{,}32\ cm$; $\tau_w \simeq 0{,}1\ \frac{N}{m^2}$