INTRODUCTION TO TURBULENCE

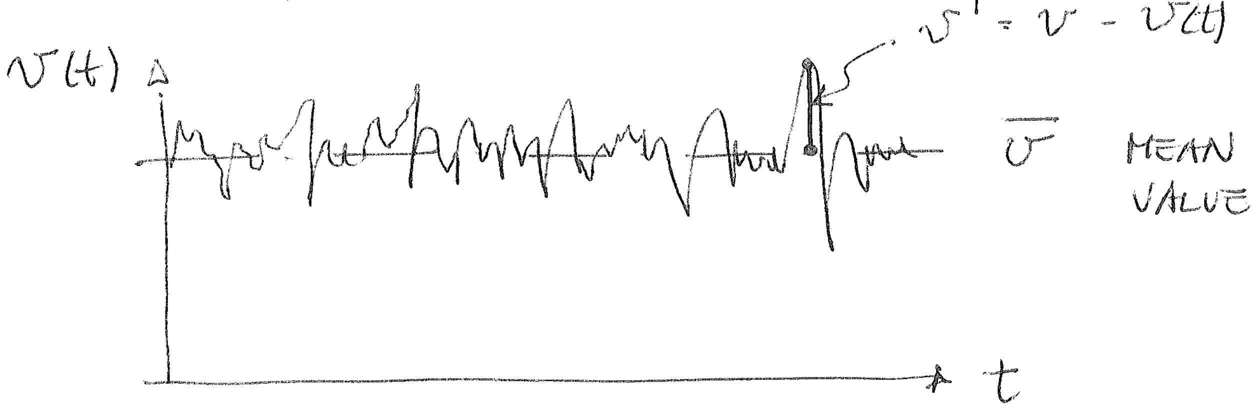
Turbulent flows are ubiquitous in nature, and all share the following main characteristics:

1. T.F. are highly unstable and time-dependent (namely unsteady)
2. T.F. are highly 3D
3. T.F. are strongly diffusive (they favour diffusive mixing of fluid particles) and dissipative* (mixing enhances momentum exchange between elementa[l] volumes of fluid: This process also leads to a loss of kinetic energy, which is irreversebly dissipat[ed] (converted) into internal fluid energy.
4. T.F. are highly rotational (fluid vorticity plays an important role)
5. T.F. are characterized by COHERENT STRUCTURES, namely events that occur with a characteristic frequency within the flow, that are characterized by spatially-coherent motion of fluid blobs, and that have an essentially-deterministic (not random) nature. These structures are responsible for all transport phenomena occurring inside the flow (mass, momentum, energy)

* even if viscosity has a wea[k] effect at large Reynolds number, it cannot be neglect[ed] and this makes turbulence comp[lex]

As a result of these characteristics, turbulent flows are usually characterized by complex velocity fields, which however can be measured/computed and characterized statistically.

Suppose you can make field measurements of wind velocity. If the wind is blowing steadily during the measurement, then the data collected will look similar to this plot:



The instantaneous velocity oscillates around a mean value (obtained as time average of the instantaneous value) yet without a specific, characteristic frequency. Therefore, velocity oscillations (FLUCTUATIONS) around the mean have a non-deterministic (random) behaviour.

Clearly, repeating the experiment at later times will produce different instantaneous values for the wind velocity, even if the mean value (and other statistical quantities like the standard deviation) could remain the same.

How can we describe such a complex velocity field? Through the usual conservation equations, namely:

(1) $\frac{\partial v_i}{\partial x_i} = 0$ CONTINUITY

(2) $\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2}$ NAVIER-STOKES

if we assume that the fluid is incompressible and Newtonian remembering that:

$$v_i(x,y,z,t) = \bar{v}_i(x,y,z) + v_i'(x,y,z,t)$$

with: $\bar{v}_i(x,y,z) \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} v_i(x,y,z,t)\,dt$

For continuity we find:

$$\frac{\partial v_i}{\partial x_i} = \frac{\partial(\bar{v}_i + v_i')}{\partial x_i} = \underbrace{\frac{\partial \bar{v}_i}{\partial x_i}}_{\substack{\text{This is always}\\ \text{equal to zero}\\ \text{due to fluid's}\\ \text{incompressibility}}} + \underbrace{\frac{\partial v_i'}{\partial x_i}}_{\substack{\text{Hence, this is}\\ \text{also equal to zero}}} = 0$$

$$\boxed{\frac{\partial v_i}{\partial x_i} = \frac{\partial \bar{v}_i}{\partial x_i} = \frac{\partial v_i'}{\partial x_i} = 0}$$

For Navier-Stokes we find:

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2}$$

$$\rho\left[\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial v_i'}{\partial t} + (\bar{v}_j + v_j')\frac{\partial(\bar{v}_i + v_i')}{\partial x_j}\right] =$$

$$= -\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial p'}{\partial x_i} + \mu \cdot \underbrace{\frac{\partial^2(\bar{v}_i + v_i')}{\partial x_j^2}}_{\mu\left(\frac{\partial^2 \bar{v}_i}{\partial x_j^2} + \frac{\partial^2 v_i'}{\partial x_j^2}\right)}$$

Now: $$(\bar{v}_j + v_j')\frac{\partial(\bar{v}_i + v_i')}{\partial x_j} =$$

$$= \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + v_j' \frac{\partial \bar{v}_i}{\partial x_j} + \bar{v}_j \frac{\partial v_i'}{\partial x_j} + v_j' \frac{\partial v_i'}{\partial x_j}$$

Let us take the time average of N.S.:

$$\overline{\rho\left[\quad \dots \quad\right]} = \overline{-\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial p'}{\partial x_i} + \mu\left(\frac{\partial^2 \bar{v}_i}{\partial x_j^2} + \frac{\partial^2 v_i'}{\partial x_j^2}\right)}$$

$$\rho\left[\underbrace{\overline{\frac{\partial \bar{v}_i}{\partial t}}}_{= \frac{\partial \bar{v}_i}{\partial t}} + \underbrace{\cancel{\overline{\frac{\partial v_i'}{\partial t}}}}_{=0} + \underbrace{\overline{\bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j}}}_{= \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j}} + \underbrace{\cancel{\overline{v_j' \frac{\partial \bar{v}_i}{\partial x_j}}}}_{=0} + \underbrace{\cancel{\overline{\bar{v}_j \frac{\partial v_i'}{\partial x_j}}}}_{=0} + \right.$$

$$+ \overline{v_j' \frac{\partial v_i'}{\partial x_j}} \Bigg] = -\underbrace{\overline{\frac{\partial \bar{p}}{\partial x_i}}}_{=\frac{\partial \bar{p}}{\partial x_i}} - \underbrace{\cancel{\overline{\frac{\partial p'}{\partial x_i}}}}_{=0} + \mu\left(\underbrace{\overline{\frac{\partial^2 \bar{v}_i}{\partial x_j^2}}}_{=\frac{\partial^2 \bar{v}_i}{\partial x_j^2}} + \underbrace{\cancel{\overline{\frac{\partial^2 v_i'}{\partial x_j^2}}}}_{=0}\right)$$

Since: $\overline{v + u} = \bar{v} + \bar{u}$

$$\bar{\bar{v}} = \bar{v}$$

$$\overline{v'} = 0 \qquad (\text{but } \overline{(v')^2} > 0 \;!!)$$

$$\overline{v \cdot u} = \overline{(\bar{v} + v') \cdot (\bar{u} + u')} =$$

$$\overset{!}{=} \bar{v} \cdot \bar{u} + \overline{v' \cdot u'}$$

$$\overline{\bar{v} \cdot u} = \bar{v} \cdot \bar{u}$$

One is left with:

$$\boxed{\rho\left(\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \overline{v_j' \frac{\partial v_i'}{\partial x_j}}\right) = -\frac{\partial \bar{p}}{\partial x_i} + \mu \frac{\partial^2 \bar{v}_i}{\partial x_j^2}} \qquad (3)$$

which is an equation similar to eq. (2) with the only differences that:

i) it is written for $\bar{v}_i$ rather than for v_i

ii) there is an extra term $\overline{v_j' \frac{\partial v_i'}{\partial x_j}}$, which is unknown!

NOTE: Subtracting eq. (3) from eq. (2) yields a momentum conservation equation for v_i':

$$\rho\left(\frac{\partial v_i'}{\partial t} + \bar{v}_j \frac{\partial v_i'}{\partial x_j} + v_j' \frac{\partial \bar{v}_i}{\partial x_j} + v_j' \frac{\partial v_i'}{\partial x_j} - \underbrace{\overline{v_j' \frac{\partial v_i'}{\partial x_j}}}_{\text{same unknown term!}}\right) =$$

$$= -\frac{\partial p'}{\partial x_i} + \mu \frac{\partial^2 v_i'}{\partial x_j^2}$$

With these equations we can describe any kind of environmental flow in the turbulent regime, provided that the fluid is incompressible and Newtonian.

To use these equations, however, one needs to find the expression for the unknown term:

$$\overline{v_j' \frac{\partial v_i'}{\partial x_j}} \underset{\text{x-component!}}{=} \underbrace{\overline{v_x' \frac{\partial v_x'}{\partial x}}}_{A} + \underbrace{\overline{v_y' \frac{\partial v_x'}{\partial y}}}_{B} + \underbrace{\overline{v_z' \frac{\partial v_x'}{\partial z}}}_{C} =$$

$$= \underbrace{\overline{\frac{\partial (v_x' v_x')}{\partial x} - v_x' \frac{\partial v_x'}{\partial x}}}_{A} + \underbrace{\overline{\frac{\partial (v_x' v_y')}{\partial y} - v_x' \frac{\partial v_y'}{\partial y}}}_{B}$$

$$+ \underbrace{\overline{\frac{\partial (v_x' v_z')}{\partial z} - v_x' \frac{\partial v_z'}{\partial z}}} =$$

$$= \overline{\frac{\partial(v_x' v_x')}{\partial x}} + \overline{\frac{\partial(v_x' v_y')}{\partial y}} + \overline{\frac{\partial(v_x' v_z')}{\partial z}}$$

$$\underbrace{- \overline{v_x' \frac{\partial v_x'}{\partial x}} - \overline{v_x' \frac{\partial v_y'}{\partial y}} - \overline{v_x' \frac{\partial v_z'}{\partial z}}}_{\cancel{+\overline{v_x' \frac{\partial v_y'}{\partial y}}} \cancel{+ \overline{v_x' \frac{\partial v_z'}{\partial z}}} \cancel{- \overline{v_x' \frac{\partial v_y'}{\partial y}}} \cancel{- \overline{v_x' \frac{\partial v_z'}{\partial z}}} = 0}$$

Since $\frac{\partial v_x'}{\partial x} = -\frac{\partial v_y'}{\partial y} - \frac{\partial v_z'}{\partial z}$ from continuity.

Thus we find :

$$\overline{v_j' \frac{\partial v_i'}{\partial x_j}} = \frac{\overline{\partial(v_i' v_j')}}{\partial x_j}$$

with eq. (3) becoming :

$$\boxed{\rho \left[\frac{\partial \overline{v}_i}{\partial t} + \overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} + \frac{\overline{\partial(v_i' v_j')}}{\partial x_j} \right] = -\frac{\partial \overline{p}}{\partial x_i} + \mu \frac{\partial^2 \overline{v}_i}{\partial x_j^2}} \quad (4)$$

This equation is called Reynolds-Averaged Navier-Stokes (RANS equation) and is the fundamental equation used to describe the evolution of the mean velocity (namely the mean flow field).

In this equation, the quantity $\rho \overline{\frac{\partial (v_i' v_j')}{\partial x_j}}$ is called REYNOLDS STRESS. More precisely, it is the ij component of the Reynolds stress tensor, which is made of 9 elements:

$$\rho \overline{\frac{\partial (v_x' v_x')}{\partial x}} \quad ;$$

$$\rho \overline{\frac{\partial (v_x' v_y')}{\partial y}} \quad ;$$

$$\rho \overline{\frac{\partial (v_x' v_z')}{\partial z}} \quad ;$$

$$\rho \overline{\frac{\partial (v_y' v_x')}{\partial x}} \quad ;$$

$$\rho \overline{\frac{\partial (v_y' v_y')}{\partial y}} \quad ;$$

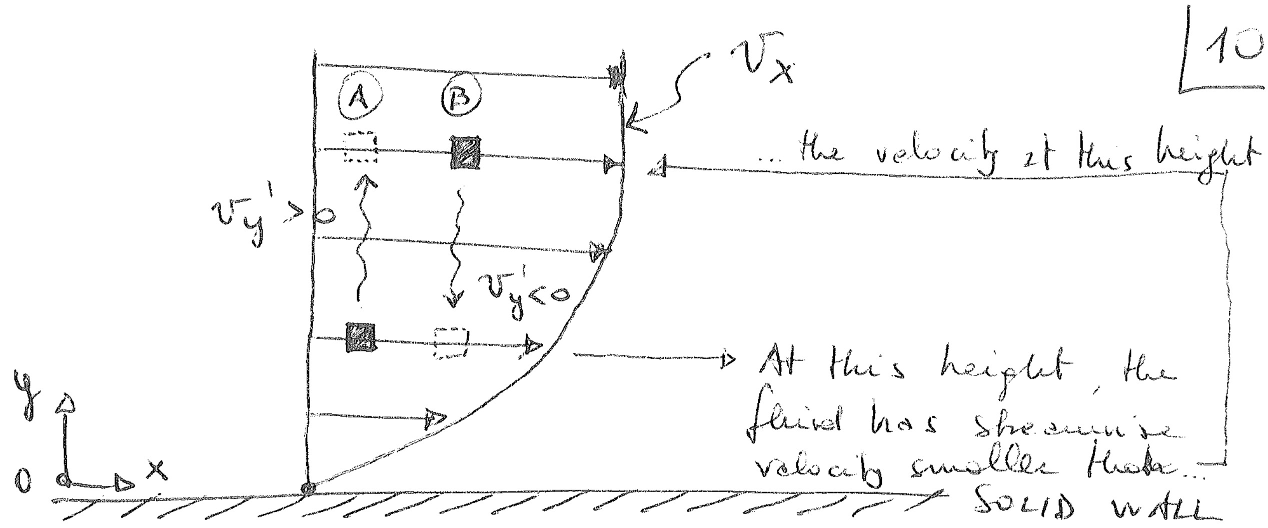
and so on...

All these terms are unknown, and add to the "original" unknowns (3 velocity components + pressure) giving a total of 13 unknowns. Yet still 4 equations available!

This system of 13 unknowns and 4 equations cannot be solved unless we are able to come up with an expression for the Reynolds stresses. The derivation of such expression is known as CLOSURE PROBLEM, and is at the heart of all modelling issues in turbulence.

How can we solve the closure problem? By deriving a suitable model for the additional unknowns, that is the Reynolds stresses. To derive such model, one can think of the physical meaning of such stresses. Consider in particular the x-y component of these stresses: if it is non zero than the fluid element subject to the stress must be subject to both v_x' and v_y'. Suppose now that y is the vertical direction, and x the horizontal direction, and that the fluid element belongs to a boundary layer flow like the one depicted in the following page:

→



Two cases are possible:

- CASE A: the fluid element is subject to a positive velocity fluctuation in the vertical direction, $v_y' > 0$. As a consequence, the element must move away from the solid wall reaching a region where the fluid moves faster in the streamwise direction: Therefore, the fluid element (upon reaching the new vertical position) will move slower than the surrounding fluid elements along x, and will experience a negative streamwise velocity fluctuation, $v_x' < 0$. What happens is that the slower fluid element tries to decelerate the faster fluid elements around it (and vice versa), thus generating an additional stress (the Reynolds stress), which adds to the viscous stress : $\rho v_x' v_y' < 0$!

- CASE B : the fluid element is subject to a negative velocity fluctuation in the vertical direction, $v_y' < 0$. As a consequence, the element must move towards the solid wall, reaching a region where the surrounding fluid moves slower in the streamwise direction : Contrary to case A, this results in a positive streamwise velocity fluctuation, $v_x' > 0$. The faster fluid element tries to accelerate the slower fluid elements around it (and viceversa), generating again a negative $\underbrace{\text{additional}}_{\text{additional to viscous stress}}$ stress, $\rho v_x' v_y' < 0$!

In both cases, the total stress acting on the fluid element is :

$$\tau_{xy}^{TOT} = \underbrace{\mu \frac{d \overline{v_x}}{dy}}_{\substack{\text{Viscous stress} \\ \text{("laminar" stress)}}} - \underbrace{\rho \overline{v_x' v_y'}}_{\substack{\text{Reynolds stress} \\ \text{(turbulent stress)}}}$$

This does not exist in laminar flow!

Recalling eq. (4):

$$\rho\left[\frac{\partial \overline{v}_i}{\partial t} + \overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j}\right] = -\frac{\partial \overline{p}}{\partial x_i} + \underbrace{\mu \frac{\partial^2 \overline{v}_i}{\partial x_j^2} - \rho \frac{\partial \overline{(v_i' v_j')}}{\partial x_j}}_{\frac{\partial}{\partial x_j}\Big[\underbrace{\mu \frac{\partial \overline{v}_i}{\partial x_j} - \rho \overline{v_i' v_j'}}_{\overline{\tau}_{ij}^{TOT}}\Big]}$$

$$= -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau}_{ij}^{TOT}}{\partial x_j} \quad (5)$$

Now, because $\rho\, v_i' v_j'$ is analogous to a stress, it is reasonable to model it as it were a viscous stress, namely:

$$\tau_{ij}^{VISC} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \rightarrow \boxed{\tau_{ij}^{TURB} = \mu^{TURB}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)} \quad (6)$$

where μ^{TURB} is known as TURBULENT VISCOSITY.

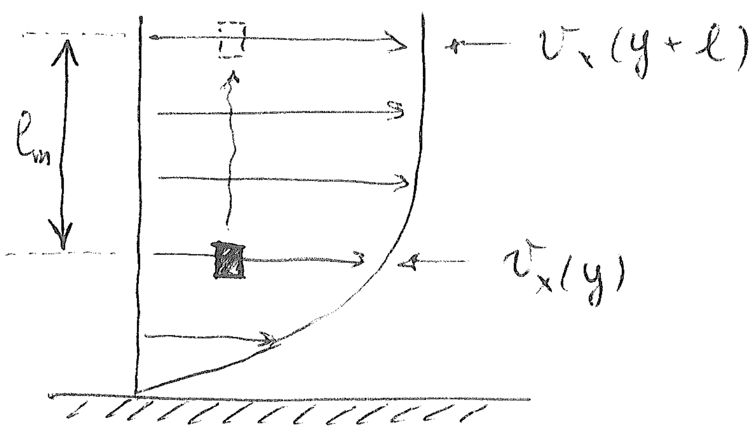
This analogy was first proposed by Boussinesq in 1877, and is based on the possibility of defining μ^{TURB} (which is clearly NOT a physical property of the fluid, but just the parameter of the model.

The basic idea is that, upon finding a suitable expression for μ^{TURB} (namely, upon finding a suitable model), it is possible to use eq. (6) into eq. (5) and solve the closure problem.

So, now the issue is: how to find a model for μ^{TURB}? The easiest way, first exploited by Prandtl in 1933, is to resort to dimensional analysis. It must be:

$$\mu^{TURB} \sim \rho \cdot l_m \cdot |\Delta v|$$

where $|\Delta v|$ is the velocity change experienced by a generic fluid element over a certain distance l_m. The quantity $|\Delta v|$ can be expressed as a function of l_m as follows:



$$\Delta v = v_x(y+l) - v_x(y)$$

$$\frac{\Delta v}{l_m} = \underbrace{\frac{v_x(y+l) - v_x(y)}{l_m}}_{= \frac{\partial v_x}{\partial y} \text{ if } l_m \to 0}$$

Hence : $|\Delta v| \sim l_m \left| \frac{\partial v_x}{\partial y} \right|$

$$\Downarrow$$

$$\boxed{\mu^{TURB} = \rho\, l_m^2 \left| \frac{\partial v_x}{\partial y} \right|}$$

PRANDTL'S MIXING LENGTH MODEL

where l_m is now defined as <u>MIXING LENGTH</u>. From a conceptual point of view, the mixing length is analogous to the concept of mean free path in thermodynamics: it is the distance over which a fluid element is able to conserve its properties (momentum, in particular) before mixing with the surrounding fluid (thus "losing" its properties). After having travelled a distance equal to l_m, the model assumes that the fluid element cannot keep its coherence anymore and is bound to mix with other fluid elements.

Clearly, Prandtl's model is very simple (and embodies many approximations). Yet, it has been widely used in many fields, including atmospheric science and oceanography.

To properly close the model, one must provide the expression for l_m: such expression depends on the type of turbulent flow and, in general, is space dependent.