Solution of Homework N^o 2: transport/storage of compressible fluids

a.

1. Pipeline geometrical data and the knowledge about the transformation of the gas as it flows along the line can be used to verify if the pressure of tank B is large enough to generate sonic flow conditions. From Bernoulli equation written for isothermal flow along a pipeline when the flow is sonic, indicated with 1 and 2 respectively the values of pressure at pipe inlet ($p_1 \simeq p_B$) and at pipe outlet in tank A, we get:

$$\ln \frac{p_1}{p_2} + \frac{1}{2}\left(1 - \left(\frac{p_1}{p_2}\right)^2\right) + 2f\frac{L}{D} = 0 \qquad (1)$$

from which we can calculate the critical pressure ratio:

$$\left(\frac{p_1}{p_2}\right)^{II} = \sqrt{1 + 4f\frac{L}{D} + \left(2\ln\left(\frac{p_1}{p_2}\right)^{I}\right)} \qquad (2)$$

Neglecting at first the logarithmic term, we get: $p_1/p_2 = 3.605$ and by iterating including the contribution of the logarithmic term we converge at $(p_1/p_2)_{cr} = 3.969$. Since $p_B/p_A = 25 \gg (p_1/p_2)_{cr}$, the gas flow is sonic. Alternatively, we can calculate the minimum value of pressure in tank B necessary to generate sonic flow along the pipeline, which is $p_{B,min,cr} = (p_1/p_2)_{cr} p_{atm} = 3.969\ atm$. Since the initial value of pressure in tank B is larger than $p_{B,min,cr}$, ten the flow is sonic. For sonic flow

- the transferred gas flow rate depends on pressure and density establishing at the outlet section of the pipe:

$$G = \sqrt{p_2\rho_2} = \sqrt{\frac{M}{RT}}p_2 \qquad (3)$$

- the critical pressure ratio between the upstream and downstream sections of the pipe is constant

$$p_2 = \left(\frac{p_1}{p_2}\right)_{cr}^{-1} p_1 = 6.299 \cdot 10^5 Pa \qquad (4)$$

Therefore we get $G(t = 0) = 2135.56\ kg/m^2s$. Transferred flow rate is given by $\dot{m} = GA = 4.19\ kg/s$.

2. The mass of gas transferred from B to A to charge tank A from 1 atm to 15 atm can be calculated as:

$$\Delta m = m_{fin} - m_{in} = M(n_{fin} - n_{in}) = \frac{MV}{RT}(p_{A,fin} - p_{A,in}) \qquad (5)$$

where the ideal gas law has been used to calculate the number of moles of gas n given the pressure. From calculations we get $m = 80.46\ kg$. It would be possible (but much more complex) to calculate the transferred mass integrating in time the value of the specific flow rate G transported along the line. The functional relationship $G = f(p_1)$ is simple until the flow remains sonic but becomes more complex to integrate when, due to gas accumulation in tank A, p_A becomes large enough to change the sonic regime into sub-sonic. This happens when $p_A = 6.299 \cdot 10^5 Pa$, i.e. the filling of the tank is made by sub-sonic flow for a significant fraction of the time (from 6.299 to 15 atm). During this period, we should integrate G given by:

$$G = \sqrt{\frac{\frac{M}{2RT}\left(p_1^2 - p_2^2\right)}{\ln\frac{p_1}{p_2} + 2\frac{fL}{D}}} \qquad (6)$$

where $p_1 = p_B = 25\ atm$ and $p_2 = f(t)$ from mass conservation in tank B.

3. Assuming isothermal efflux from tank A, to check if the flow is sonic we need to compare the pressure in the tank with the value necessary to produce sonic flow. Indicating with p_A the value of pressure in tank A and with p_3 the value of pressure at the broken section, Bernoulli equation written for efflux from the tank (neglecting viscous losses) under isothermal condition and sonic flow gives:

$$\left(\frac{p_A}{p_3}\right)_{cr} = e^{1/2} = 1.649 \qquad (7)$$

alternatively we can calculate the minimum value of pressure in tank A required to generate sonic flow, $p_{A,min,cr} = (p_A/p_3)_{cr} p_{atm} = 1.649 \cdot 10^5 Pa$. Being $p_A = 20 \cdot 10^5 Pa$, the flow is sonic at starting time and until $p_A > p_{A,min,cr}$ is true.

4. Closing the flow from tank B, the mass balance on tank A gives:

$$\frac{MV}{RT}\frac{\mathrm{d}p_A}{dt} = -A_r G \qquad (8)$$

where A_r is the hole section and G is the specific flow rate, which under sonic condition is:

$$G = \sqrt{p_3\rho_3} = \sqrt{\frac{M}{RT}}p_3 = \sqrt{\frac{M}{RT}}\left(\frac{p_A}{p_3}\right)_{cr}^{-1} p_A = k p_A \qquad (9)$$

with $k = 0.0021$ and

$$\frac{\mathrm{d}p_A}{dt} = -\frac{RT}{MV} k p_A = -K_T p_A \qquad (10)$$

with $K_T = 0.0176\ s^{-1}$. Separating the variables and integrating between the values of p_A at starting time and at the final time (t^*) of sonic flow, we get

$$t^* = \frac{1}{K_T} \ln \frac{p_A(0)}{p_A(t^*)} = 142\ s \tag{11}$$

where $p_A(t^*) = p_{A,min,cr}$.

b.

1. Pipeline geometrical data and knowledge about the transformation of the gas as it flows from the tank to the burner can be used to verify if the pressure in the tank is large enough to generate sonic flow conditions. Using 1 and 2 to indicate respectively the upstream ($p_1 \simeq p_{serb}$) and downstream section of the pipe, Bernoulli equation written for isothermal flow along a pipeline when the flow is sonic gives

$$\ln \frac{p_1}{p_2} + \frac{1}{2}\left(1 - \left(\frac{p_1}{p_2}\right)^2\right) + 2f\frac{L}{D} = 0 \tag{12}$$

from which we can calculate the critical pressure ratio:

$$\left(\frac{p_1}{p_2}\right)^{II} = \sqrt{1 + 4f\frac{L}{D} + \left(2\ln\left(\frac{p_1}{p_2}\right)^{I}\right)} \tag{13}$$

Neglecting at first the logarithmic term, we get $p_1/p_2 = 9.539$ and by subsequent iterations $(p_1/p_2)_{cr} = 9.775$. Since $p_{serb}/p_{atm} = 30 \gg (p_1/p_2)_{cr}$, the flow is sonic. Alternatively, we can calculate the minimum pressure in the tank required to generate sonic flow, which is: $p_{1,min,cr} = (p_1/p_2)_{cr} p_{atm} = 9.775\ atm$. Since at starting time the pressure in the tank ($30\ atm$) is larger than $p_{1,min,cr}$, the flow is sonic. For sonic flow

- the flow rate depends on pressure and density establishing at the downstream section

$$G = \sqrt{p_2 \rho_2} = \sqrt{\frac{M}{RT}} p_2 \tag{14}$$

- the ratio between upstream and downstream pressure is constant and equal to the critical pressure ratio

$$p_2 = \left(\frac{p_1}{p_2}\right)_{cr}^{-1} p_1 = 3.069 \cdot 10^5 Pa \tag{15}$$

We calculate $G(t = 0) = 1259.38\ kg/m^2 s$. The mass flow rate is given by $\dot{m} = GA = 39.56\ kg/s$.

2. Even when the pressure in the tank halves, its value is large enough to generate sonic flow. The mass conservation equation written for the gas in the tank gives:

$$\frac{MV}{RT}\frac{\mathrm{d}p_1}{dt} = -AG \tag{16}$$

where A is the section of the pipe and G is the specific flow rate, given by

$$G = \sqrt{p_2 \rho_2} = \sqrt{\frac{M}{RT}} p_2 = \sqrt{\frac{M}{RT}} \left(\frac{p_1}{p_2}\right)_{cr}^{-1} p_1 = k p_1 \tag{17}$$

from which

$$\frac{\mathrm{d}p_1}{dt} = -\frac{RT}{MV} k p_1 = -K_T p_1 \tag{18}$$

with $K_T = 0.01958\ s^{-1}$. Separating the variables and integrating between the starting ($30\ atm$) and final ($15\ atm$) values of pressure in the tank we get

$$t = \frac{1}{K_T} \ln \frac{p_1(0)}{0.5 p_1(0)} = \frac{1}{K_T} \ln 2 = 35.4\ s \tag{19}$$

3. The mass of gas transferred to the burner up to that time can be calculated as

$$\Delta m = m_{in} - m_{fin} = M(n_{in} - n_{fin}) = \frac{MV}{RT}(p_{1,in} - p_{1,fin}) \tag{20}$$

where the ideal gas law has been used to evaluate the number of moles of the gas n associated to each value of pressure. From calculations we get $m = 1010.34\ kg$.

c.

1. Indicating with 0 and 1 respectively the value of pressure inside the tank and at the outlet section of valve V_1, for sonic efflux from the tank under isothermal condition, the critical pressure ratio is $p_1/p_2 = e^{1/2} = 1.649$. The minim value of pressure in the tank required to generate sonic flow is $p_{1,min,cr} = (p_1/p_2)_{cr} p_{atm} = 1.649\ atm$. Being the starting value of pressure larger than this value, the flux through the valve will be sonic.

2. The mass conservation equation written for the gas in the tank is

$$\frac{MV}{RT}\frac{\mathrm{d}p_1}{dt} = -A_v G \tag{21}$$

where A_v is the valve cross-section. Supposing that the flow is sonic up to 40 s, G is

$$G = \sqrt{p_2 \rho_2} = \sqrt{\frac{M}{RT}} p_2 = \sqrt{\frac{M}{RT}} \left(\frac{p_1}{p_2}\right)_{cr}^{-1} p_1 = k p_1 \tag{22}$$

from which

$$\frac{\mathrm{d}p_1}{dt} = -\frac{RT}{MV} k p_1 = -K_T p_1 \tag{23}$$

where $K_T = 0.00743\ s^{-1}$. Separating the variables and integrating we can calculate the value of pressure after 40 s:

$$p_1(t = 40\ s) = p_1(0)\exp(-K_T t) = 7.428 \cdot 10^5 Pa \quad (24)$$

Since this value is larger than $p_{1,min,cr}$, the initial assumption (sonic flow during 40 s) was correct. The mass of gas transferred in this time is

$$\Delta m = m_{in} - m_{fin} = M(n_{in} - n_{fin}) = \frac{MV}{RT}(p_{1,in} - p_{1,fin}) \quad (25)$$

where the ideal gas law has been used to calculate the number of moles of gas n for each value of pressure. From calculations we get $m = 16.89\ kg$.

3. When the valve V_2 opens, the pressure in the tank is equal to $7.428 \cdot 10^5 Pa$. This pressure value could be large to generate sonic flow along the pipeline. Indicating with p_3 the pressure at the downstream section of the pipe, the critical pressure ratio (isothermal Bernoulli equation for the pipeline under sonic flow) is given by

$$\ln\frac{p_1}{p_3} + \frac{1}{2}\left(1 - \left(\frac{p_1}{p_3}\right)^2\right) + 2f\frac{L}{D} = 0 \quad (26)$$

from which we get the critical pressure ratio

$$\left(\frac{p_1}{p_3}\right)^{II} = \sqrt{1 + 4f\frac{L}{D} + \left(2\ln\left(\frac{p_1}{p_3}\right)^{I}\right)} \quad (27)$$

Neglecting (initially) the logarithmic term, we get $p_1/p_3 = 11$ and by successive iterations $(p_1/p_3)_{cr} = 11.217$. Since $p_1/p_{atm} = 7.428 \ll (p_1/p_3)_{cr}$, the flow is sub-sonic. Using isothermal Bernoulli equation for the pipeline under sub-sonic conditions we get

$$G = \sqrt{\frac{(p_1^1 - p_{atm}^2)M/(2RT)}{\ln(p_1/p_{atm}) + 2fL/D}} = 169.4\ kg/m^2 s \quad (28)$$

The mass flow rate is $\dot{m} = GA = 0.083\ kg/s$.

d.

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1. The mass conservation equation for the gas

$$\frac{MV}{RT}\frac{\mathrm{d}p_1}{dt} = w_{in} - w_{out} = w_{in} - GA \quad (29)$$

where w_{in} and w_{out} represent the in-going and out-going flow rates. At steady state conditions, $\mathrm{d}p/\mathrm{d}t = 0$ and the out-going and in-going mass flow rates should be equal. Indicating with A the cross section of the pipe we get $G = 318.30\ kg/m^2 s$. To calculate the pressure in the tank we need to check if the flow is sonic or not. We can calculate the minimum value of flow transferred under sonic flow as:

$$G_{min,cr} = \sqrt{p_{atm}\rho_{atm}} = \sqrt{\frac{M}{RT}}p_{atm} = 256.28\ kg/m^2 s \quad (30)$$

Pressure and density in the outlet section become equal to environmental values when the flow regime changes from sonic to subsonic. Since $G > G_{min,cr}$, the flow is sonic. The value of pressure at the downstream section is fixed by the flow rate G and is given by:

$$G_{cr} = \sqrt{p_2\rho_2} = \sqrt{\frac{M}{RT}}p_2 \rightarrow p_2 = G\sqrt{\frac{RT}{M}} = 1.242 \cdot 10^5 Pa \quad (31)$$

Since the flow is sonic, the value of pressure at the upstream section is fixed by the critical pressure ratio, which depends on the pipeline geometry and friction factor f. Using Blasius equation to calculate the friction factor we get $f = 0.079Re^{-0.25}$ where $Re = GD/\mu$ and $f = 0.002166$. For the pipeline under investigation we get (isothermal Bernoulli equation for a pipeline under sonic flow)

$$\ln\frac{p_1}{p_2} + \frac{1}{2}\left(1 - \left(\frac{p_1}{p_2}\right)^2\right) + 2f\frac{L}{D} = 0 \quad (32)$$

from which we can calculate the critical pressure ratio

$$\left(\frac{p_1}{p_2}\right)^{II} = \sqrt{1 + 4f\frac{L}{D} + \left(2\ln\left(\frac{p_1}{p_2}\right)^{I}\right)} \quad (33)$$

Neglecting at staring time the logarithmic term we get $p_1/p_2 = 8.386$ and iterating $(p_1/p_2)_{cr} = 8.639$. We can then calculate $p_1 = (p_1/p_2)_{cr}p_2 = 10.73 \cdot 10^5 Pa$.

2. During maintenance operations, the mass balance equation for the gas in the tank is:

$$\frac{MV}{RT}\frac{\mathrm{d}p_1}{dt} = w_{in} \quad (34)$$

where $w_{in} = cost$. The pressure variation is linear in time:

$$p_1(t) = p_1(0) + \frac{w_{in}RT}{MV}t \quad (35)$$

and we can calculate the time at which $p_1(t) = 10\ atm$:

$$t = \frac{p_1(t) - p_1(0)}{w_{in}RT/MV} = 24.356\ s \quad (36)$$

3. When the valve opens, the tank which is at $p_1 = 20\ atm$ discharges gas into the environment. Assuming adiabatic efflux from the tank the critical

pressure ratio (Bernoulli equation for adiabatic efflux from a tank under sonic condition, $\gamma = 1.3$) is:

$$\left(\frac{p_1}{p_2}\right)_{cr} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} = 1.83 \tag{37}$$

The minimum value of pressure generating sonic flow from the tank is $p_{1,min,cr} = p_1/p_2)_{cr} p_{atm} = 1.83 \cdot 10^5\ Pa$. Since $p_1 > p_{1,min,cr}$, the flow is sonic and the specific mass flow rate is given by:

$$G = \sqrt{\gamma p_2 \rho_2} = \sqrt{\gamma \frac{M}{RT_2}} p_2 = \tag{38}$$

$$= \sqrt{\gamma \frac{M}{RT_2}} \left(\frac{p_1}{p_2}\right)_{cr}^{-1} p_1 \tag{39}$$

We can calculate T_2 from T_1 using the relationship $T^\gamma/p^{\gamma-1}$:

$$\left(\frac{T_1}{T_2}\right)_{cr} = \left(\frac{p_1}{p_2}\right)_{cr}^{\frac{\gamma-1}{\gamma}} = \frac{\gamma+1}{2} = 1.15 \tag{40}$$

from which $T_2 = 254.8\ K$. Substituting T_2 in 39 we get $G = 3481.74\ kg/m^2 s$.

e.

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1. Each segment of the natural gas pipeline starts with a recompression unit. Indicating with 1 and 2 respectively the point upstream and downstream the recompression unit, and with 3 the end of the pipeline (3=1 for a periodic line), the minimum pressure along the line is at point 3. We should check that $p_1 = p_3 \geq 1.5 \cdot 10^5\ Pa$. The mass flow rate transferred along the pipeline is:

$$\dot{m} = Q\rho \tag{41}$$

where ρ s the gas density at measuring conditions (i.e. pressure and temperature at which that volumetric flow rate has been measured). From ideal gas law $\rho = p_{ref} M/RT_{ref} = 0.66\ kg/m^3$ and $\dot{m} = 23.\ kg/s$. Being the pipe cross section known, $G = \dot{m}/A = 325.38\ kg/m^2 s$. The point of minimum pressure is at compressor inlet (and/or at the end of the line). Therefore we impose $p_3 = p_{min} = 1.5 \cdot 10^5 Pa$, and using Bernoulli equation for isothermal flow along a pipeline under non sonic conditions we can calculate p_2:

$$p_2 = \sqrt{p_3^2 + \frac{2flG^2}{D}\frac{2RT}{M}} = 16.14 \cdot 10^5 Pa \tag{42}$$

Pressure at compressor outlet will be $p_2 = 16.14 \cdot 10^5 Pa$.

2. If the pipeline breaks, the gas inside the pipe at the breaking point (pressurized at $1.5 \cdot 10^5\ Pa$) will be free to flow toward the outer environment, which is at atmospheric pressure. Assuming adiabatic flow through the broken section (adiabatic efflux of gas from a tank) we can calculate the critical pressure ratio (between the pressure in inside the pipe at the breaking point, p_3, and the pressure at the broken section, p_r), which is given by:

$$\left(\frac{p_3}{p_r}\right)_{cr} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} = 1.89 \tag{43}$$

The minimum pressure required to generate sonic flow from the broken section is $p_{3,min,cr} = 1.89 \cdot 10^5 Pa$. Since the pressure in the pipe is less than this value, the flow is sub-sonic. The specific flow rate is given by

$$G = \rho_1 v_1 = \rho_1 \sqrt{\frac{2\gamma}{\gamma-1}\left[\frac{p_3}{\rho_3} - \frac{p_{env}}{\rho_{env}}\right]} \tag{44}$$

where $\rho_3 = p_3 M/RT$, $p_{env} = 1\ atm$ and $\rho_{env} = p_{env} M/RT_r$ with T_r gas temperature at the broken section, given by the adiabatic law. Since the gas expands from p_3, T to p_{env}, T_r, we get $p_{env}^{\gamma-1}/T_r^\gamma = p_3^{\gamma-1}/T^\gamma$. For methane (polyatomic gas) is $\gamma = 1.33$ and calculations give $T_r = T(p_3/p_{env})^{(1-\gamma)/\gamma} = 0.91 \cdot T = 266\ K$, $\rho_3 = 0.72\ kg/m^3$ from which we can calculate$G$.

f.

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1. Mass conservation for the gas in the tank is

$$\frac{MV}{RT}\frac{\mathrm{d}p_s}{dt} = M\dot{n}(t) = M\dot{n}_0 \exp[kt] \tag{45}$$

where the term on the RHS is the mass of gas generated by the chemical reaction. This equation can be integrated in time from the starting condition $p_s(0) = 2\ atm$, to derive the evolution of gas pressure inside the tank:

$$p_s(t) = p_s(0) + \frac{\dot{n}_0 RT}{Vk}(\exp(kt) - 1) \tag{46}$$

which increases exponentially. To calculate the time before the valve opens we need to isolate t

$$t = \frac{1}{k}\ln\left(1 + \frac{p_s(t) - p_s(0)}{\dot{n}_0 RT} Vk\right) \tag{47}$$

From problem data we get $t = 18.26\ s$.

2. When the valve opens, the mass balance for the gas in the tank becomes

$$\frac{MV}{RT}\frac{\mathrm{d}p_s}{dt} = M\dot{n}(t) = M\dot{n}_0 \exp[kt] - AG \tag{48}$$

where G is the specific flow rate exiting from the safety valve. Considering adiabatic efflux from the tank, the critical pressure ratio between the pressure inside the tank, p_s, and the pressure at the outlet section, p_o, is

$$\left(\frac{p_s}{p_o}\right)_{cr} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} = 1.89 \tag{49}$$

The minimum pressure value required to produce sonic flow is $p_{s,min,cr} = 1.89 \cdot 10^5 Pa$. Pressure inside the tank at the time at which the valve opens is $15\ atm > p_{s,min,cr}$ therefore the flow is sonic. The specific flow rate exiting from the valve is given by

$$G = \sqrt{\gamma p_o \rho_o} = \sqrt{\gamma \frac{M}{RT_o}} p_o \tag{50}$$

where $p_o^{\gamma-1} T_o^{\gamma} = p_s^{\gamma-1} T_s^{\gamma}$ from the adiabatic transformation which using the critical pressure ration we get $T_s/T_o) = (p_s/p_o)_{cr}^{(\gamma-1)/\gamma} = (\gamma+1)/2 = 1.2$. When the valve opens we have $p_o = 7.936 \cdot 10^5 Pa$ and $T_o = 250\ K$ from which $G = 3190.77\ kg/m^2 s$. The mass flow rate exiting from the valve is $\dot{m} = GA = 1.59\ kg/s$. The pressure variation inside the tank after the valve opens, depends on the balance between the two mass flow rates which are opposite in sign: the flux of gas exiting from the valve contributes to pressure reduction, whereas the chemical reaction contributes to pressure increase. Based on the mass balance we have:

$$\frac{MV}{RT}\frac{\mathrm{d}p}{\mathrm{d}t} = M\dot{n}_0 \exp[kt] - AK_1 p \tag{51}$$

which should be integrated in t from $p = 15\ atm$ when the valve opens. The differential equation is of the type:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = A_1 \exp[kt] - A_2 p \tag{52}$$

and its solution is the sum of one term which is exponentially increasing (produced by the chemical reaction) and one term which is exponentially decreasing, due to the emptying of the tank. The solution is

$$p(t) = C_1 \exp[kt] + C_2 \exp[-A_2 t] \tag{53}$$

where C_1 and C_2 are constants to be calculated. Deriving the general solution we get

$$\frac{p(t)}{\mathrm{d}t} = C_1 k \exp[kt] - C_2 A_2 \exp[-A_2 t] = A_1 \exp[kt] - A_2(C_1 \exp[kt] + C_2 \exp[-A_2 t]) \tag{54}$$

Equating terms containing $\exp[kt]$ we get:

$$C_1 k = A_1 - A_2 C_1 \rightarrow C_1 = A_1/(k + A_2) \tag{55}$$

C_2 can be evaluated imposing the starting value of pressure ($C_2 = p(0) - A_1/(k + A_2)$). The exponentially growing term prevails in the short time, leading to an indefinite pressure increase inside the tank which can only be delayed by the opening of the valve.

g.

1. For adiabatic efflux from the tank, the critical pressure ratio between the pressure in the tank, p_s, and the pressure at the outlet section, p_o, is given by:

$$\left(\frac{p_s}{p_o}\right)_{cr} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} = 1.83 \qquad per\ \ \gamma = 1.33 \tag{56}$$

   The minimum pressure value required to have sonic flow is $p_{s,min,cr} = 1.83 \cdot 10^5 Pa$. Since the value of pressure inside the tank when the efflux starts is $10\ atm > p_{s,min,cr}$, the flow is sonic.

2. The specific flow rate exiting from the tank is given by

$$G = \sqrt{\gamma p_o \rho_o} = \sqrt{\gamma \frac{M}{RT_o}} p_o \tag{57}$$

   where $p_o^{\gamma-1} T_o^{\gamma} = p_s^{\gamma-1} T_s^{\gamma}$ from the adiabatic transformation and $(T_s/T_o) = (p_s/p_o)_{cr}^{(\gamma-1)/\gamma} = (\gamma+1)/2 = 1.165$ using the critical pressure ratio. We get $p_o = 5.46 \cdot 10^5 Pa$ and $T_o = 251.5\ K$ from which $G = 1510.35\ kg/m^2 s$. The mass flow rate exiting from the tank is $\dot{m} = 0.474\ kg/s$.

3. Even when the pressure inside the tank decreases to 3 atm, the flow is sonic. The mass conservation for the gas in the tank is:

$$\frac{MV}{RT}\frac{\mathrm{d}p_s}{dt} = -GA = -A\sqrt{\gamma \frac{M}{RT_o}} p_o = \tag{58}$$

$$= -A\sqrt{\gamma p_s \rho_s \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{59}$$

   from which

$$\frac{\mathrm{d}p_s}{\mathrm{d}t} = -\frac{A}{V}\sqrt{\gamma \frac{RT_o}{M}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\, p_s = -K_T p_s \tag{60}$$

   with $K_T = 0.008467$. Separating the variables and integrating we get

$$p_s(t) = p_s(0)\left(1 - \exp(-K_T t)\right) \tag{61}$$

   from which we can calculate the time t:

$$t = \frac{1}{k}\ln\frac{p_s(0)}{p_s(t)} = 142\ s \tag{62}$$

4. The mass of gas discharged from the tank while the pressure is changing from 10 atm to 3 atm is given by

$$\Delta m = M(n_i - n_f) = \frac{MV}{RT_s}(p_{s,i} - p_{s,f}) = 45.98\ kg \tag{63}$$

h.

1. The flow rate to be transferred corresponds to a specific flow rate equal to $G = w/A = 407.43\ kg/m^2s$. Indicating with 1 the pressure at the upstream section of the pipeline and with 2 the pressure at the outlet section of the line, connected with tank B, the minimum flow rate which can be transported as sonic flow is given by:

$$G_{min,cr} = \sqrt{p_2 \rho_2} = \sqrt{\frac{M}{RT}} p_2 \tag{64}$$

where p_2 is (at minimum) that of the outer environment, p_{atm}. We get $G_{min,cr} = 359.38\ kg/m^2s$. Since $G > G_{min,cr}$ the flow transferred will be sonic.

2. The value of pressure at the outlet section is given by

$$p_2 = \frac{G}{\sqrt{M/RT}} = 1.133 \cdot 10^5 Pa \tag{65}$$

3. The pressure in the tank, which is the same as p_1, is calculated considering the critical pressure ratio established between the upstream and downstream sections of the pipe when the flow is sonic. From isothermal Bernoulli equation from a pipeline under sonic condition we get

$$\ln \frac{p_1}{p_2} + \frac{1}{2}\left(1 - \left(\frac{p_1}{p_2}\right)^2\right) + 2f\frac{L}{D} = 0 \tag{66}$$

from which we can calculate the critical pressure ratio

$$\left(\frac{p_1}{p_2}\right)^{II} = \sqrt{1 + 4f\frac{L}{D} + \left(2\ln\left(\frac{p_1}{p_2}\right)^{I}\right)} \tag{67}$$

Since G is known, we can calculate f using Blasius equation, $f = 0.079Re^{-0.25} = 0.0024$. Neglecting (initially) the logarithmic term we get $p_1/p_2 = 7.655$ and iterating $(p_1/p_2)_{cr} = 7.921$. Since the downstream pressure is known, the upstream pressure is calculated as $p_1 = (p_1/p_2)_{cr} p_2 = 8.974 \cdot 10^5 Pa$.

4. If a leakage is produced in the tank, assuming isothermal flow we have $p_A/p_o = e^{0.5} = 1.648$. The minimum value of pressure in tank required to generate sonic flow is $p_{A,min,cr} = 1.648 \cdot 10^5 Pa$. Since when the leakage is produced is $p_A = p_1 = 8.974 \cdot 10^5 Pa > p_{A,min,cr}$, the flow is sonic. For sonic isothermal flow $G = \sqrt{p_o \rho_o} = \sqrt{M/RT} p_o$ where $p_o = (p_1/p_2)_{cr}^{-1} p_A = 5.44 \cdot 10^5\ Pa$, and we get $G = 1955.06\ kg/m^2s$ and $\dot{m} = GA = 0.614\ kg/s$.

i.

Assuming that pressure losses at the inlet (1) and at the outlet (2) compared to frictional losses along the line, $p_A \sim p_1$ and $p_2 \sim p_B$.

1. If viscous losses are negligible, Bernoulli equation is:

$$\frac{1}{2}dv^2 + \frac{dp}{\rho} + gdh = 0 \tag{68}$$

which can be integrated as

$$\frac{v_2^2}{2} - \frac{v_1^2}{2} + g(h_2 - h_1) + \frac{RT}{M}\ln\frac{p_B}{p_A} = 0 \tag{69}$$

where $v_2 = G/\rho_2$ and $v_1 = G/\rho_1$ with ρ_1 and ρ_2 calculated from the value of pressure in the tanks and the value of T along the line using the gas law:

$$\begin{aligned} &\frac{G^2}{2}\left(\frac{RT}{M}\right)^2\left(\frac{1}{p_B^2} - \frac{1}{p_A^2}\right) + \\ &+ g(h_2 - h_1) + \frac{RT}{M}\ln\frac{p_B}{p_A} = 0 \end{aligned} \tag{70}$$

from which G can be calculated.

2. If the viscous losses are comparable to gravitational losses, Bernoulli equation becomes

$$\frac{1}{2}dv^2 + \frac{dp}{\rho} + gdh = -2\frac{f}{D}v^2dx \tag{71}$$

Since $dx = dh/cos\alpha$, being the variation of gas velocity along the pipe unknown, we need to write v as a function of G, which does not change along the pipe. We get:

$$-G^2\frac{\mathrm{d}\rho}{\rho} + \rho\mathrm{d}p + 2\frac{f}{D}G^2dx + \rho^2 g\mathrm{d}h = 0 \tag{72}$$

Substituting dx and dp we get:

$$\left(-\frac{G^2}{\rho} + \rho\frac{RT}{M}\right)d\rho = -\left(\frac{2fG^2}{D\cos\alpha} + \rho^2 g\right)dh \tag{73}$$

and by variable separation we get:

$$\frac{-G^2/\rho + \rho RT/M}{2fG^2/D\cos\alpha + \rho^2 g}\mathrm{d}\rho = -\mathrm{d}h$$

or

$$f(\rho) \cdot \mathrm{d}\rho = -\mathrm{d}h \quad . \tag{74}$$

The integral can be solved decomposing the function $f(\rho)$ into factors, being the degree of the numerator less than the denominator. This function, once integrated, allows to calculate the value of G transferred between the two tanks.

j.

1. To calculate the flow rate exiting at starting time from the well, we need to find the value of the specific flow rate G. The pressure in the well is high enough to suppose sonic flow condition, nevertheless is possible to check if this is true a priori considering that:

   - since the line is long, the pressure drop between the tank outlet and the pipe inlet is negligible compared to the pressure drop along the line ($p_0 \simeq p_1$);
   - for isothermal flow and sonic flow, the critical pressure ratio between the pressure upstream, p_1, and downstream the line, p_2, is given by

   $$\ln\frac{p_1}{p_2} + \frac{1}{2}\left(1 - \left(\frac{p_1}{p_2}\right)^2\right) + 2f\frac{L}{D} = 0 \tag{75}$$

   This equation can be iteratively solved if geometrical data of the pipeline are given assuming a value for the friction coefficient f (e.g. $f = 0.003$)

   $$\left(\frac{p_1}{p_2}\right)^I = \sqrt{1 + 4f\frac{L}{D}} \tag{76}$$

   $$\left(\frac{p_1}{p_2}\right)^{II} = \sqrt{1 + 4f\frac{L}{D} + \left(2\ln\left(\frac{p_1}{p_2}\right)^I\right)} \tag{77}$$

   which after a few iterations gives $(p_1/p_2)_{cr} = 10.08$. Using this value we can check is the flow is sonic at starting time, since the minimum value of pressure in the well required to generate sonic flow is $p_{0,min,crit} = 10.08\ p_{atm} = 10.08 \cdot 10^5 Pa$. At starting time we have $p_0 = 25 \cdot 10^5 Pa > p_{0,min,crit}$, therefore the flow is sonic.

   Since the flow is sonic, the specific flow rate exiting from the well depends on conditions establishing at the pipeline outlet section, which differ from those in the outer environment but are linked to the pressure inside the well by the critical pressure ratio. We get

   $$G = \sqrt{p_2\rho_2} = \sqrt{\frac{M}{RT}p_2^2} = \sqrt{\frac{M}{RT}}p_2 \tag{78}$$

   where the ideal gas law has been used to express ρ_2 and p_2 at the outlet section, and we used the critical pressure ratio to relate the pressure upstream and downstream the pipe:

   $$G = \sqrt{\frac{M}{RT}}\left(\frac{p_1}{p_2}\right)^{-1}_{crit} p_1 = K_1 p_1 \tag{79}$$

   From this equation we calculate $G = 635.62\ kg/m^2s$, and the mass flow rate is $\dot{m} = G \cdot A = 4.99\ kg/s$.

2. The pressure inside the well when the well starts discharging natural gas is given by

   $$p_2(0) = \left(\frac{p_1}{p_2}\right)^{-1}_{cr} p_1(0) = 2.48\ atm \tag{80}$$

   which is larger than the value required to transfer the desired flow rate. The laminarization valve installed on the line will be used to generate additional pressure loss reducing the pressure at the end of the line from 2.48 atm to the desired value until this will be necessary, i.e. until the pressure inside the well will decrease to the minimum value of pressure able to transfer the desired flow rate. The time to exhaustion of the well is given by the value of pressure for which gas flowing to the well head reaches $1.5 \cdot 10^5$ without valve regulation (laminarization valve completely open). The specific flow rate in this condition is subsonic and given by:

   $$G_{min,crit} = \sqrt{\frac{M}{RT}}p_{atm} = 256.28\ kg/s \tag{81}$$

   and the pressure at the upstream section of the pipe is (using isothermal Bernoulli equation along a pipeline under non sonic condition)

   $$G^2\ln\frac{p_1}{p_2} + \frac{M}{2RT}\left(p_2^2 - p_1^2\right) + \frac{2fL}{D}G^2 = 0 \tag{82}$$

   Neglecting the logarithmic term and solving for p_1 being $p_2 = 1.5\ atm$ and $G = 180\ kg/s$ known. This value of p_1^* identify the last time at which the desired flow rate can be extracted from the well. Assuming that the flow rate exiting from the well has been continuously regulated by the valve to the desired constant value, the time to exhaustion of the well is given by:

   $$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\frac{RT}{MV}\frac{\pi d^2}{4}G \tag{83}$$

   $$\int_{p_1(0)}^{p_1^*}\mathrm{d}p_1 = -\frac{RT}{MV}\frac{\pi D^2}{4}Gt^* \tag{84}$$

   from which we get

   $$t^* = \frac{4MV(p_1(0) - p_1^*)}{\pi D^2 GRT} \tag{85}$$

k.

1. The mass balance for the gas contained inside the tank gives

   $$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{VM}{RT}\frac{\mathrm{d}p_0}{\mathrm{d}t} = -AG \tag{86}$$

This equation can be integrated once the right expression for G is selected, depending on the flow regime (sonic or sub-sonic). The pressure inside the tank ($1\ MPa = 10\ atm$) seems to be large enough to assume sonic flow. However, we can check a priori if this is true: for isothermal flow along a line and sonic condition, the critical pressure ratio between the pressure upstream, $p_1 \simeq p_0$, and downstream the line, p_2, is given by

$$\ln\frac{p_1}{p_2} + \frac{1}{2}\left(1 - \left(\frac{p_1}{p_2}\right)^2\right) + 2f\frac{L}{D} = 0 \tag{87}$$

This equation can be solved iteratively if we know the pipeline geometry and assume a trial value for the friction factor f (e.g. $f = 0.003$):

$$\frac{p_1}{p_2}^{I\ tent} = \sqrt{1 + 4f\frac{L}{D}} \tag{88}$$

$$\frac{p_1}{p_2}^{II\ tent} = \sqrt{1 + 4f\frac{L}{D} + 2\ln\frac{p_1}{p_2}^{I\ tent}} \tag{89}$$

$$\frac{p_1}{p_2}^{III\ tent} = \sqrt{1 + 4f\frac{L}{D} + 2\ln\frac{p_1}{p_2}^{II\ tent}} \tag{90}$$

$$... \tag{91}$$

$$\tag{92}$$

After a few iterations we get

$$\frac{p_1}{p_2}_{crit} = 3.97 \tag{93}$$

which indicates that the flow will be sonic until the pressure inside the tank is larger (or equal) than $p_1^* = 3.97p_{atm}$. This value gives the lower integration limit necessary to find the time f duration of the sonic flow. To integrate the equation 97 we should use the right expression for G which is:

$$G = \sqrt{p_1\rho_1} \tag{94}$$

where p_2 and ρ_2 are conditions establishing at the outlet section of the pipe, which are linked to conditions at the upstream section of the line through the critical pressure ratio,

$$G = \sqrt{\frac{M}{RT}p_2^2} = \sqrt{\frac{M}{RT}}p_2 = \tag{95}$$

$$= \sqrt{\frac{M}{RT}}\left(\frac{p_1}{p_2}\right)_{crit}^{-1} p_1 = K_1 p_1 \tag{96}$$

The mass conservation equation becomes

$$\frac{VM}{RT}\frac{\mathrm{d}p_1}{\mathrm{d}t} = -AK_1p_1 \tag{97}$$

which can be integrated as

$$\int_{p_1(0)}^{p_1^*}\frac{\mathrm{d}p_1}{p_1} = \frac{AK_1RT}{MV}t \tag{98}$$

from which

$$t = \ln\frac{p_1(0)}{p_1^*}\frac{MV}{AK_1RT} \tag{99}$$

2. If the tank discharges directly into the environment, the specific flow rate G changes because conditions establishing at the outlet section changes. Specifically, starting again from Bernoulli equation written for the emptying tank assuming isothermal transformation, we get that at sonic flow condition

$$\left(\frac{p_2}{p_1}\right)_{crit} = \ln 2 = 0.69 \tag{100}$$

In the mass conservation equation we need to change:

- the upper integration limit $p_1^* = p_{atm}/0.69$
- the value f the constant K_1, which includes the new value of the critical pressure ratio.