Compressible Flow along pipelines

a.

Consider a pipe (diameter D, lenght L) used to transport gas (molar mass M, ratio of specific heat evaluated at constant pressure and at constant volume given by $\gamma = c_p/c_v$) from point 1 to point 2.

1. Derive the relation between specific mass flow rate G and pressure at the upstream/downstream section of the pipeline. Assume isothermal transformation for the gas ($p/\rho = C$) along the pipeline.
   Since the pipeline works in steady state conditions, $\dot{m} = cost$ and if the pipe diameter is constant along the length of pipe, also $G = const$. The flow along the line is described by Benrnoulli equation:

$$\mathrm{d}\frac{v^2}{2} + \frac{\mathrm{d}p}{\rho} + g\mathrm{d}h = \mathrm{d}w_s - \frac{\mathrm{d}l_v}{\rho} \tag{1}$$

where $g\mathrm{d}h \simeq 0$, $\mathrm{d}w_s = 0$ (no compressor along the line) and $\mathrm{d}l_v/\rho = 2f\mathrm{d}xv^2/D$ as for incompressible flow. Bernoulli should be integrated between point 1 and 2, but it can not be integrated in the form of Equation 1 because we do not know the law of variation of velocity along the pipe, $v(x)$, which is required to quantify the viscous losses. To integrate that term, we need to rewrite Bernoulli as:

$$\rho^2\left(\mathrm{d}\frac{v^2}{2} + \frac{\mathrm{d}p}{\rho}\right) = -\frac{2f(v^2\rho^2)\mathrm{d}x}{D} \tag{2}$$

which, considering $G = \rho v$, becomes

$$\frac{\rho^2 G^2}{2}\mathrm{d}\frac{1}{\rho^2} + \rho\mathrm{d}p = -\frac{2fG^2\mathrm{d}x}{D} \tag{3}$$

Integration of the first term gives

$$\int_1^2 \frac{\rho^2G^2}{2}\mathrm{d}\frac{1}{\rho^2} = \int_1^2 \frac{\rho^2G^2}{2}(-2)\frac{1}{\rho^3}\mathrm{d}\rho = -G^2\frac{\mathrm{d}\rho}{\rho} = -G^2\ln\frac{\rho_2}{\rho_1} = -G^2\ln\frac{p_2}{p_1} = G^2\ln\frac{p_1}{p_2} \tag{4}$$

where we considered the relationship between p and ρ in isothermal flow. Integration of the second term gives

$$\int_1^2 \rho\mathrm{d}p = \int_1^2 \frac{pM}{RT}\mathrm{d}p = \frac{M}{2RT}(p_2^2 - p_1^2) \tag{5}$$

Integration of the right hand side gives

$$\int_1^2 \frac{2fG^2\mathrm{d}x}{D} = 2f\frac{L}{D}G^2 \tag{6}$$

The integral version of Bernoulli equation for isothermal flow along the pipeline is:

$$G^2\ln\frac{p_1}{p_2} + \frac{M}{2RT}(p_2^2 - p_1^2) + 2f\frac{L}{D}G^2 = 0 = F(G, p_1, p_2) \tag{7}$$

from which we calculate

$$G = \sqrt{\frac{\frac{M}{2RT}(p_1^2 - p_2^2)}{\ln\frac{p_1}{p_2} + 2f\frac{L}{D}}} = f(p_1, p_2) \tag{8}$$

the relationship we were looking for. If we assume that the upstream pressure p_1 is fixed, the specific flow rate is a function of the downstream pressure p_2 only. p_2 can varies in the range $[0 : p_1]$. G is positive definite, and is null for both $p_2 = p_1$ and $p_2 = 0$. For the Weierstass theorem it has maximum in the range $[0 : p_1]$. To find the maximu we should calculate $\mathrm{d}G(p_2)/\mathrm{d}p_2 = 0$. Since the function $G(p_2)$ is rather complex, we can use Equation 7 to derive the derivative of G in a simpler way. Considering the differential of $F(G, p_1, p_2)$ we get

$$\mathrm{d}F(G,p_2) = \frac{\partial F}{\partial G}\mathrm{d}G + \frac{\partial F}{\partial p_2}\mathrm{d}p_2 = 0 \rightarrow \frac{\mathrm{d}G}{\mathrm{d}p_2} = -\frac{\partial F/\partial p_2}{\partial F/\partial G} \tag{9}$$

from which $\frac{\mathrm{d}G}{\mathrm{d}p_2} = 0$ only if $\partial F/\partial p_2 = 0$. Therefore:

$$\frac{\partial F}{\partial p_2} = -\frac{G^2}{p_2} + \frac{M}{RT}p_2 = 0 \rightarrow G = \sqrt{\frac{M}{RT}p_2^2} = \sqrt{\rho_2 p_2} \tag{10}$$

and the velocity of gas at the outlet section is $v_2 = G/\rho_2 = \sqrt{p_2/\rho_2} = v_{sound}$. When G is maximum, the velocity of gas at the outlet section is the sound speed (evaluated in isothermal conditions). Substituting the value of G_{max} in Equation 7 we obtain

$$\frac{M}{RT}p_2^2 \ln\frac{p_1}{p_2} + \frac{Mp_2^2}{2RT}\left[1 - \left(\frac{p_1}{p_2}\right)^2\right] + 2f\frac{L}{D}\frac{M}{RT}p_2^2 = 0 \tag{11}$$

which can be simplified as

$$\ln\frac{p_1}{p_2} + \frac{1}{2}\left[1 - \left(\frac{p_1}{p_2}\right)^2\right] + 2f\frac{L}{D} = 0 \tag{12}$$

This is a funtion of (p_1/p_2) only and gives the value of the critical pressure ratio corresponding to $G = G_{max}$. As already discussed for the efflux from a tank, when the gas velocity at the outlet section equals the speed of sound, if the outer pressure is lowered there is no way for this information to propagate upstream along the pipe toward the inlet section. Therefore, once the critical flow is established, no further variation of G is expected even if p_2 is reduced and $G = G_{max}$.

2. Derive the formula for the specific mass flow rate at critical conditions when the gas undergoes adiabatic (irreversible, i.e. non isoentropic) transformation moving along the pipe.

In this case, the integral form of Bernoulli will be given by:

$$G^2 \ln\frac{p_1}{p_2} + \int_1^2 \rho \mathrm{d}p + 2f\frac{L}{D}G^2 = 0 \tag{13}$$

where we need a relationship between ρ and p along the pipeline which is valid for irreversible adiabatic transformation. We cannot use $p/\rho^\gamma = cost$ which is only valid for revesible adiabatic. Nevertheless, we can derive an alternative relationship from the total energy conservation equation:

$$\mathrm{d}(e + \frac{p}{\rho} + \frac{1}{2}v^2 + gh) = \mathrm{d}q + \mathrm{d}w_s \tag{14}$$

e is the internal energy, q is the heat flux added to the control volume of gas and w_s is the mechanical work made on the control volume. If we define the enthalpy as $H = e + p/\rho$, the total energy equation becomes

$$\mathrm{d}(H + \frac{1}{2}v^2 + gh) = \mathrm{d}q + \mathrm{d}w_s \tag{15}$$

where we can neglect gh, $\mathrm{d}w_s = 0$ (there is no compressor on the pipeline) and $\mathrm{d}q = 0$ (for adiabatic flow). We obtain

$$\mathrm{d}(H + \frac{1}{2}v^2) = 0 \rightarrow c_p\mathrm{d}T + \frac{\mathrm{d}v^2}{2} = 0 \tag{16}$$

where, according to Mayer equation, c_p is given by

$$c_p = \frac{\gamma}{\gamma - 1}\frac{R}{M} \tag{17}$$

and according to ideal gas law $\mathrm{d}T = M/R\mathrm{d}(p/\rho)$ and

$$\frac{\gamma}{\gamma-1}\mathrm{d}\left(\frac{p}{\rho}\right)+\frac{1}{2}\mathrm{d}v^2=0 \tag{18}$$

$$\frac{2\gamma}{\gamma-1}\left(\frac{p}{\rho}\right)+\frac{G^2}{\rho^2}=C=cost \tag{19}$$

Equation 19 gives a relationship between p and ρ which is valid for each point along the pipe and can be used to integrate the term in Equation 13. From 19 we caan write

$$p=\left(C-\frac{G^2}{\rho^2}\right)\frac{\gamma-1}{2\gamma}\rho \tag{20}$$

and

$$\mathrm{d}p=\left[-(-2)\frac{G^2}{\rho^3}\rho+\left(C-\frac{G^2}{\rho^2}\right)\right]\frac{\gamma-1}{2\gamma}\mathrm{d}\rho=\left[\frac{2G^2}{\rho^2}+C-\frac{G^2}{\rho^2}\right]\frac{\gamma-1}{2\gamma}\mathrm{d}\rho=\left[\frac{G^2}{\rho^2}+C\right]\frac{\gamma-1}{2\gamma}\mathrm{d}\rho \tag{21}$$

The integral in Bernoulli becomes

$$\int_1^2\rho\mathrm{d}p=\int_1^2\rho\left[\frac{G^2}{\rho^2}+C\right]\frac{\gamma-1}{2\gamma}\mathrm{d}\rho=\frac{\gamma-1}{2\gamma}\left[G^2\ln\frac{\rho_2}{\rho_1}+\frac{C}{2}(\rho_2^2-\rho_1^2)\right] \tag{22}$$

Further simplifications led to

$$\int_1^2\rho\mathrm{d}p=\frac{\gamma-1}{2\gamma}\left[G^2\ln\frac{\rho_2}{\rho_1}+\left(\frac{\gamma}{\gamma-1}p_1\rho_1+\frac{G^2}{2}\right)\left(\left(\frac{\rho_2}{\rho_1}\right)^2-1\right)\right] \tag{23}$$

and the final form for integral Bernoulli equation for adiabatic irreversible flow:

$$G^2\ln\frac{\rho_1}{\rho_2}\frac{\gamma+1}{\gamma}+\left(p_1\rho_1+\frac{\gamma-1}{\gamma}G^2\right)\left[\left(\frac{\rho_2}{\rho_1}\right)^2-1\right]+4f\frac{L}{D}G^2=0 \tag{24}$$

which is again an implicit function $F(G,\rho_1,\rho_2)=0$. If we consider the upstream condition fixed, $F(G,\rho_2)=0$ describes the variation of G as the downstream condition change. Following the same procedure as before, we can find the condition for which $G=G_{max}$ from

$$\frac{\mathrm{d}G}{\mathrm{d}\rho_2}=0\rightarrow\frac{\mathrm{d}G}{\mathrm{d}\rho_2}=-\frac{\partial F/\mathrm{d}\rho_2}{\partial F/\partial G}\rightarrow\frac{\partial F}{\mathrm{d}\rho_2}=0 \tag{25}$$

which gives

$$\frac{\partial F}{\mathrm{d}\rho_2}=-\frac{G^2}{\rho_2}\frac{\gamma+1}{\gamma}+\left(\rho_1p_1+\frac{\gamma-1}{2\gamma}G^2\right)\frac{2}{\rho_1^2}\rho_2=0 \tag{26}$$

and using Equation 19 to rewrite the term in parentesis, we get

$$\frac{G^2}{\rho_2}\frac{\gamma+1}{\gamma}=\frac{\gamma-1}{2\gamma}\left[\frac{G^2}{\rho_2^2}+\frac{2\gamma}{\gamma-1}\frac{p_2}{\rho_2}\right]2\rho_2 \tag{27}$$

and

$$\frac{G^2}{\rho_2}\left[\frac{\gamma+1}{\gamma}-\frac{\gamma-1}{\gamma}\right]=2p_2\rightarrow2\frac{G^2}{\gamma}=2\rho_2p_2\rightarrow G_{max}=\sqrt{\gamma p_2\rho_2} \tag{28}$$

The last equation indicates that when the specific flow rate is maximum, G is a function of pressure and density at the outlet section and the formula is the same as that calculated for the flow exiting from a tank under adiabatic conditions. The velocity of gas at the outlet section turns out to be $v_2 = G/\rho_2 = \sqrt{\gamma p_2/\rho_2}$ which is the speed of sound. Substituting the value of G_{max} into Equation 24 we obtain the equation which gives the link between densities at the upstream and downstream section of the pipe at critical flow conditions:

$$\ln\frac{\rho_1}{\rho_2} + \frac{1}{2}\left[1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right] + 4f\frac{L}{D}\frac{\gamma}{\gamma+1} = 0 \tag{29}$$

which together with

$$\frac{p_1}{p_2} = \frac{\rho_1}{\rho_2}\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\frac{\rho_2}{\rho_1} \tag{30}$$

allows to link upstream and downstream conditions of the pipe. Equation 29 is the analogous of the critical pressure ratio calculated for isothermal transport. Equation 30 is derived from Equation 19 using the result of Equation 29.