Sub-critical and critical flow conditions

a.

Consider a tank (volume V, pressure p_0 and temperature T_0) filled with gas (molar mass M, ratio of specific heat evaluated at constant pressure and at constant volume given by $\gamma = c_p/c_v$) able to discharge the gas trough a nozzle (diameter d) to a receiving environment (outer pressure p_{env}).

1. Derive the relation between pressure inside the tank and in the outer environment and specific flow rate G discharged from the nozzle if viscous losses can be neglected. Assume reversible abiabatic expansion for the gas ($p/\rho^\gamma = C$).
   Efflux of gas from the tank is described by mass conservation:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \dot{m}_{in} - \dot{m}_{out} \tag{1}$$

where $\dot{m}_{in} = 0$ in this case. The mass of gas inside the tank is given by:

$$m = n_0 M = M\frac{p_0 V}{RT_0} \tag{2}$$

where n_0 is the number of moles which, according to the ideal gas law, is a function of pressure (p_0), temperature (T_0) and volume of the tank through the universal gas constant $R = 8314\ J/Kkmole$. Assuming $T_0 = const$ inside the tank (the volume and heat capacity are large enough to be unaffected by thermal variations induced by the efflux of gas), the variation of mass becomes:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{MV}{RT}\frac{\mathrm{d}p_0}{\mathrm{d}t} = -\dot{m}_{out} \tag{3}$$

where the outgoing mass can be conveniently expressed as $\dot{m}_{out} = A \cdot G$ where $G = \rho_1 v_1$ is the specific mass flux at the nozzle section and $A = \pi d^2/4$ is the nozzle area. We can use Bernoulli equation written in differential form:

$$\mathrm{d}\left(\frac{v^2}{2}\right) + g\mathrm{d}h + \frac{\mathrm{d}p}{\rho} = \mathrm{d}w_s - \frac{\mathrm{d}l_v}{\rho} \tag{4}$$

which once integrated between a position 0 inside the tank and the nozzle section 1, gives:

$$\frac{v_1^2}{2} + \int_0^1 \frac{\mathrm{d}p}{\rho} = 0 \tag{5}$$

To simplify Bernoulli equation we assumed that in the system considered (i) the gravitational term is negligible (there is no significant change of density with height between point 0 and 1 in the system considered); (ii) no energy is given to the fluid; (iii) negligible viscous losses; (iv) negligible gas velocity inside the tank. To solve the integral for the pressure term, we write $\rho = (p/C)^{1/\gamma}$ according to the adiabatic transformation we assume to hold between point 0 and 1:

$$\int_0^1 \frac{\mathrm{d}p}{\rho} = \int_0^1 \frac{C^{1/\gamma}}{p^{1/\gamma}}\mathrm{d}p = C^{1/\gamma}\int_0^1 p^{-1/\gamma}\mathrm{d}p = C^{1/\gamma}\frac{1}{-\frac{1}{\gamma}+1}\left[p^{-\frac{1}{\gamma}+1}\right]_0^1 = C^{1/\gamma}\frac{\gamma}{\gamma-1}\left[p_1^{\frac{\gamma-1}{\gamma}} - p_0^{\frac{\gamma-1}{\gamma}}\right] \tag{6}$$

Considering the value of $C^{1/\gamma} = p_1^{1/\gamma}/\rho_1 = p_0^{1/\gamma}/\rho_0$, we get

$$\int_0^1 \frac{\mathrm{d}p}{\rho} = \frac{\gamma}{\gamma-1}\left[\left(\frac{p_1^{1/\gamma}}{\rho_1}\right)p_1^{\frac{\gamma-1}{\gamma}} - \left(\frac{p_0^{1/\gamma}}{\rho_0}\right)p_0^{\frac{\gamma-1}{\gamma}}\right] = \frac{\gamma}{\gamma-1}\left[\frac{p_1}{\rho_1} - \frac{p_0}{\rho_0}\right] \tag{7}$$

from which we calculate:

$$v_1 = \sqrt{\frac{2\gamma}{\gamma-1}\left[\frac{p_0}{\rho_0} - \frac{p_1}{\rho_1}\right]} \tag{8}$$

The specific mass flow rate is given by:

$$G = \rho_1 v_1 = \sqrt{\frac{2\gamma}{\gamma-1}\rho_1^2\left[\frac{p_0}{\rho_0} - \frac{p_1}{\rho_1}\right]} \tag{9}$$

and if we use again the relation $\rho_1 = (p_1/C)^{1/\gamma}$, we get:

$$G(p_0, p_1) = \sqrt{\frac{2\gamma}{\gamma-1}\left(\frac{p_1}{C}\right)^{2/\gamma}\left[\frac{p_0}{\rho_0} - \frac{p_1}{\rho_1}\right]} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{1}{C^{2/\gamma}}\left[\frac{p_0}{\rho_0}p_1^{2/\gamma} - p_1^{2/\gamma}\frac{p_1}{p_1^{1/\gamma}}C^{1/\gamma}\right]} = \tag{10}$$

$$= \sqrt{\frac{2\gamma}{\gamma-1}\frac{1}{C^{2/\gamma}}\left[\frac{p_0}{\rho_0}p_1^{2/\gamma} - C^{1/\gamma}p_1^{1+2/\gamma-1/\gamma}\right]} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{1}{C^{2/\gamma}}\left[\frac{p_0}{\rho_0}p_1^{2/\gamma} - C^{1/\gamma}p_1^{1+1/\gamma}\right]} \tag{11}$$

If we consider p_0 in the tank fixed, the specific mass flow rate is a function of p_1 only. The pressure at the nozzle exit can change in the range $[p_{env} : p_0]$ with p_{env} arbitrary small (up to 0). In general, we can assume that the pressure at the nozzle exit p_1 will not be different from p_{env}: any variation of pressure in the outer environment propagates at the speed of sound in any surrounding direction. If $p_{env} = p_1 = p_0$ (no pressure difference between one point inside and one point outside the tank) $G(p_1) = 0$ (no flow through the nozzle). If $p_{env} = p_1 = 0$, $G(p_1) = 0$ since $\rho_1 = 0$ at $p_1 = 0$. Therefore, $G(p_1)$ is positive definite in the range, and has null values at the upper and lower end of the interval. According to Weirstrass theorem, $G(p_1$ has a maximum in $p_1 = [0 : p_0]$. To find this maximum we derive $G(p_1)$ with respect to p_1:

$$\frac{\mathrm{d}G(p_1)}{\mathrm{d}p_1} = 0 \tag{12}$$

or, which is the same, we can derive the function of p_1 which is the argument of the square root:

$$\frac{\mathrm{d}}{\mathrm{d}p_1}\left[\frac{p_0}{\rho_0}p_1^{2/\gamma} - C^{1/\gamma}p_1^{1+1/\gamma}\right] = 0 \tag{13}$$

which gives

$$\frac{p_0}{\rho_0}\frac{2}{\gamma}p_1^{2/\gamma-1} - C^{1/\gamma}\frac{1+\gamma}{\gamma}p_1^{1+1/\gamma-1} = \frac{p_0}{\rho_0}\frac{2}{\gamma}p_1^{\frac{2-\gamma}{\gamma}} - C^{1/\gamma}\frac{1+\gamma}{\gamma}p_1^{1/\gamma} = 0 \tag{14}$$

and finally

$$\frac{p_0}{\rho_0}\frac{2}{\gamma}p_1^{\frac{2-\gamma-1}{\gamma}} = C^{1/\gamma}\frac{1+\gamma}{\gamma} \tag{15}$$

$$\frac{p_0}{\rho_0}\frac{2}{\gamma}p_1^{\frac{1-\gamma}{\gamma}} = \left(\frac{p_0^{1/\gamma}}{\rho_0}\right)\frac{1+\gamma}{\gamma} \tag{16}$$

and after further simplification

$$\frac{2}{1+\gamma}p_1^{\frac{1-\gamma}{\gamma}} = p_0^{\frac{1-\gamma}{\gamma}} \rightarrow \left(\frac{p_1}{p_0}\right)_{CRIT} = \left(\frac{1+\gamma}{2}\right)^{\frac{\gamma}{1-\gamma}} = \left(\frac{2}{1+\gamma}\right)^{\frac{\gamma}{\gamma-1}} \tag{17}$$

$(p_1/p_0)_{CRIT}$ identifies the condition in which G is maximum; it is function of γ only and therefore is a constant value. For a bi-atomic gas, $\gamma = 1.4$ and $(p_1/p_0)_{CRIT} = 0.528$. For a mono-atomic gas, $\gamma = 5/3$ and $(p_1/p_0)_{CRIT} = 0.487$.

At this condition, the velocity at the nozzle is given by:

$$v_1 = \sqrt{\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1 - \frac{p_1}{p_0}\frac{\rho_0}{\rho_1}\right]} \tag{18}$$

and substituting the pressure ratio and density ratio

$$\frac{p_1}{p_0} = \left(\frac{2}{1+\gamma}\right)^{\frac{\gamma}{\gamma-1}} \text{ and } \frac{\rho_1}{\rho_0} = \left(\frac{p_1}{p_0}\right)^{1/\gamma} = \left(\frac{2}{1+\gamma}\right)^{\frac{1}{\gamma-1}} \tag{19}$$

we obtain:

$$v_1 = \sqrt{\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1 - \left(\frac{2}{1+\gamma}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{2}{1+\gamma}\right)^{\frac{-1}{\gamma-1}}\right]} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1 - \frac{2}{1+\gamma}\right]} = \tag{20}$$

$$= \sqrt{\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left(\frac{\gamma-1}{\gamma+1}\right)} = \sqrt{\frac{2\gamma}{\gamma+1}\frac{p_0}{\rho_0}} = \sqrt{\frac{2\gamma}{\gamma+1}\frac{RT_0}{M}} \tag{21}$$

Considering that for the adiabatic transformation we have also a relationship between temperature and pressure:

$$C = \frac{p}{\rho^\gamma} = \frac{p^\gamma}{\rho^\gamma}p^{1-\gamma} = \left(\frac{RT}{M}\right)^\gamma p^{1-\gamma} \rightarrow C' = \frac{T^\gamma}{p^{\gamma-1}} \tag{22}$$

we can relate T_1 with T_0 inside the tank

$$T_0 = T_1\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{2}{1+\gamma}\right)^{-\frac{\gamma}{\gamma-1}\frac{\gamma-1}{\gamma}} = \frac{1+\gamma}{2} \tag{23}$$

to rewrite the velocity at nozzle outlet as:

$$v_1 = \sqrt{\frac{2\gamma}{\gamma+1}\frac{RT_1}{M}\frac{1+\gamma}{2}} = \sqrt{\gamma\frac{RT_1}{M}} \tag{24}$$

The right end side of Equation 24 is the speed of sound at adiabatic conditions evaluated at the nozzle outlet. Since the velocity of the gas at nozzle outlet becomes equal to the sound speed when G is maximum, we conclude that if we reduce further p_{env}, there is no possibility for the pressure wave carrying this information of “pressure decrease” to propagate deep into the tank because the gas is moving out of the tank at the same velocity. No further change in the pressure at the nozzle outlet will be produced by a reduction in p_{env} and the value of G will maintain equal to the maximum.

2. Derive the formula for the specific mass flow rate at critical conditions as a function of pressure inside the tank.

   From Equation 24 we calculate

$$G = \rho_1 v_1 = \sqrt{\gamma\frac{RT_1}{M}\rho_1^2} = \sqrt{\gamma p_1 \rho_1} = \sqrt{\gamma\frac{M}{RT_1}p_1^2} = \sqrt{\gamma\frac{M}{RT_1}}p_1 \tag{25}$$

   Using Equation 17 for p_1 we end up with

$$G = \sqrt{\gamma\frac{M}{RT_1}}\left(\frac{p_1}{p_0}\right)_{CRIT} p_0 = k \cdot p_0 \tag{26}$$

   and using Equation 23 we find

$$G = \sqrt{\gamma\frac{M}{RT_0}\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}}_{CRIT} \cdot \left(\frac{p_1}{p_0}\right)^2_{CRIT}} p_0 = \sqrt{\gamma\frac{M}{RT_0}\left(\frac{p_1}{p_0}\right)^{2-\frac{\gamma-1}{\gamma}}_{CRIT}} \cdot p_0 = \sqrt{\gamma\frac{M}{RT_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \cdot p_0 \tag{27}$$

   where $k = f(T_0)$. Under critical conditions:

- the specific mass flow rate is (only) function of T_0, p_0 inside the tank;
- G is linearly proportional to the pressure inside the tank, p_0;
- the pressure at the nozzle outlet p_1 is proportional to the pressure inside the tank, p_0:

$$p_1 = \left(\frac{p_1}{p_0}\right)_{CRIT} p_0 \tag{28}$$

  and can be different (larger) than the pressure in the outer environment p_{env};
- downstream the nozzle outlet, the gas expands from p_1 to p_{env} in the outer environment.

3. Derive the relation between pressure inside the tank and in the outer environment and specific flow rate G discharged from the nozzle if viscous losses can be neglected. Assume isothermic transformation for the gas ($p/\rho = C$) between one point inside the tank and the nozzle outlet. Which is the critical pressure ratio in this case?

4. Derive the relation between pressure inside the tank and in the outer environment and specific flow rate G discharged from the nozzle if viscous losses can be neglected. Assume polytropic transformation for the gas ($p/\rho^k = C$) between one point inside the tank and the nozzle outlet. Which is the critical pressure ratio in this case?